Lecture 9: June 23, 2014.

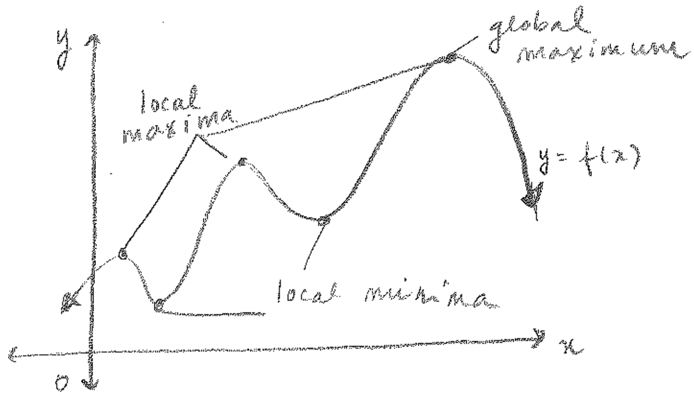
TODAY: 4.4: The First Derivative test, classifying critical points

4.5: Simple curve sketching

Recall: Let $c \in D$, where D is the domain of a fn. f. Then:

- $f(c)$ is a <u>local ("relative") maximum</u> of f when, for all x in $\underbrace{\text{some open interval containing } c}_{\text{"some neighborhood of } c\text{"}}$, $f(x) \leq f(c)$

- $f(c)$ is a <u>global ("absolute") maximum</u> of f when, for all x in D, $f(x) \leq f(c)$.



Similar for minima:

- $f(c)$ is a <u>local ("relative") minimum</u> of f when, for all x in some open interval containing c, $f(x) \geq f(c)$

- $f(c)$ is a <u>global ("absolute") minimum</u> of f when, for all x in D, $f(x) \geq f(c)$.

Also recall our procedure for finding extrema (maxima & minima):

1. Differentiate $f(x)$
2. Find critical points — i.e., where either

   (a) $f'(x) = 0$

   or

   (b) $f'(x)$ is undefined
3. Test the sign of $\boxed{f'(x)}$ on the INTERVALS between the critical points — this includes testing the limiting behavior of f — its behavior as $|x| \to \infty$.

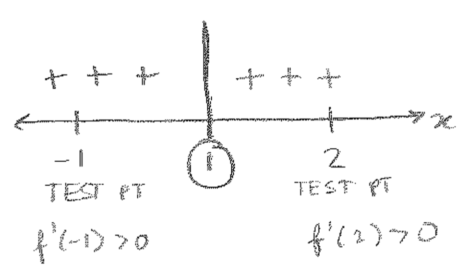
Example. Find the local and global extrema of the function

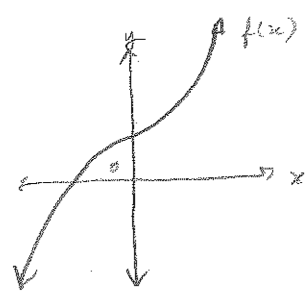
$$f(x) = x^3 - 3x^2 + 3x + 5.$$

Well, $f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$, which is never undefined, and which is zero precisely when $x = 1$. So $x = 1$ is the only critical point of f.

Options for $x = 1$:

1. local max
2. global max
3. local min
4. global min
5. neither





So $x = 1$ is not an extremum of f —

It is exactly this process that we refer to when we want to "classify the critical points" of a function.

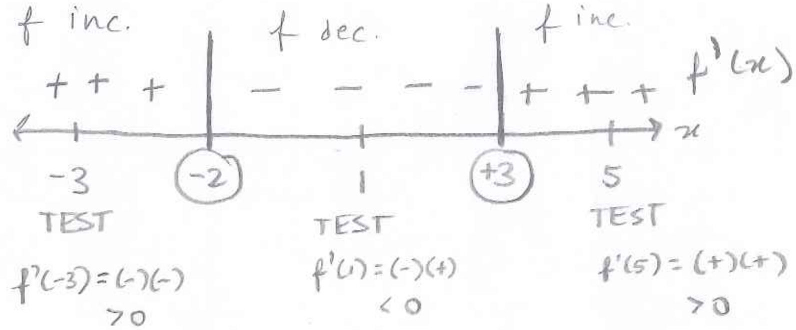
Example 1, p. 248 Find and classify the critical points of

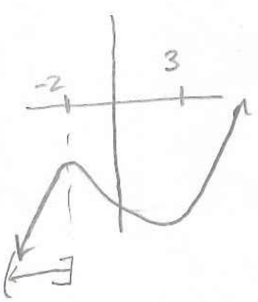
$$f(x) = 2x^3 - 3x^2 - 36x + 7.$$

Well, $f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x-3)(x+2)$, which is never undefined, and which is zero when $x = 3$ or when $x = -2$.

So the critical points of f are $x = -2$ and $x = 3$.

Let's classify them:





So, since the sign of $f'(x)$ changes from positive to negative at $x = -2$, we know that this c.p. is a local maximum — but it is NOT a global maximum, since on $(3, +\infty)$, $f'(x) > 0$, which implies $\lim_{x \to +\infty} f(x) = +\infty$ — so there is no global maximum. **MUST CHECK LIMITING BEHAVIOR!**

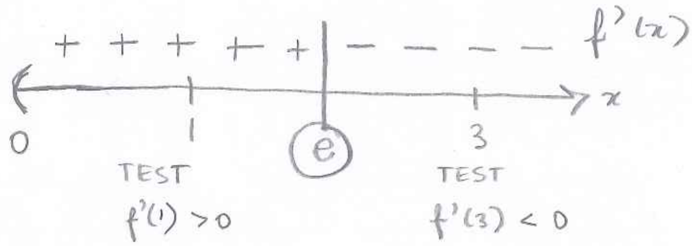
Similarly, $x = 3$ is a local minimum, but is NOT a global minimum, since $f'(x) > 0$ on $(-\infty, -2)$, which implies $\lim_{x \to -\infty} f(x) = -\infty$.

Example. 2, p. 250 $f(x) = \dfrac{2\ln(x)}{x}$ is defined on $(0, +\infty)$. Find its global extrema.

$$f'(x) = 2\frac{d}{dx}\left[\frac{1}{x}\right]\ln(x) + \frac{2}{x}\frac{d}{dx}\left[\ln(x)\right] = \frac{-2\ln(x)}{x^2} + \frac{2}{x^2} = \frac{2(1-\ln(x))}{x^2}.$$

Thus, $f'(x)$ is not defined at $x=0$ – but $x=0$ wasn't in the domain of f anyway, so disregard – and $f'(x)=0$ when $1-\ln(x)=0$ – that is, when $x=e$. (Recall, $\ln(a)=b \Leftrightarrow e^b = a$.)

So we test the sign of $f'(x)$ on either side of $x=e$:



So, $x=e$ gives a local max of f, because the sign of f' changes from positive to negative at $x=e$. To see whether this is a global max, we check limiting behavior –

A GENERAL RULE

but $f(x)$ decreases on $(e, +\infty)$ and increases on $(0, e)$, so at no point $x \neq e$ could we have $f(x) \geq f(e)$.

$x=e$ thus is the global max. of f – and f has no global minimum.

This method works for word problems as well :

Example 27, p. 253 Determine two real numbers with difference 20 and minimum possible product.

Call these numbers x and y, and note that $x - y = 20$ implies $y = x - 20$. We seek to minimize their product, $p(x) = x(x-20)$.

We differentiate first:

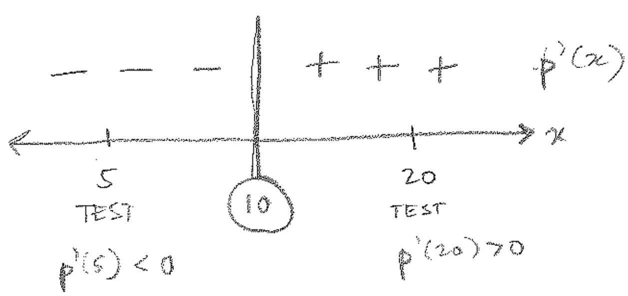
$$p'(x) = \frac{d}{dx}[x](x-20) + x\frac{d}{dx}[x-20] = (x-20) + x = 2x - 20 = 2(x-10).$$

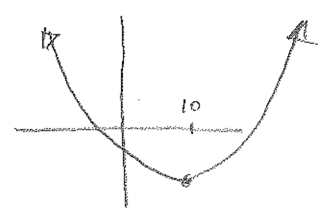
This derivative is never undefined $\longleftarrow$ MUST STATE !

and is zero when $x - 10 = 0$, i.e., when $x = 10$.

So the single critical point of p is at $x = 10$.

We hope this is a global minimum, but let's check :





Indeed, $x = 20$ gives a local minimum — also a global min., since p was decreasing on $(-\infty, 10)$ and increasing on $(10, +\infty)$.

$\longrightarrow$ MUST STATE ! (You will lose exam points if you call something a global extremum without mentioning

4.5: Curve Sketching (simple)

We now have the tools we need to construct rudimentary graphs of functions —

- Critical points of f
  - VERTICAL / HORIZONTAL TAN. LINES
- Increasing / decreasing behavior
- Limiting behavior

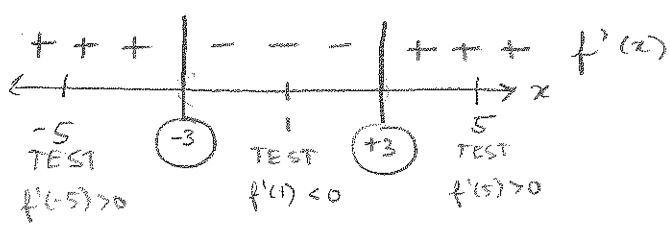
And, can also plot the x- and/or y-intercept(s) to help.

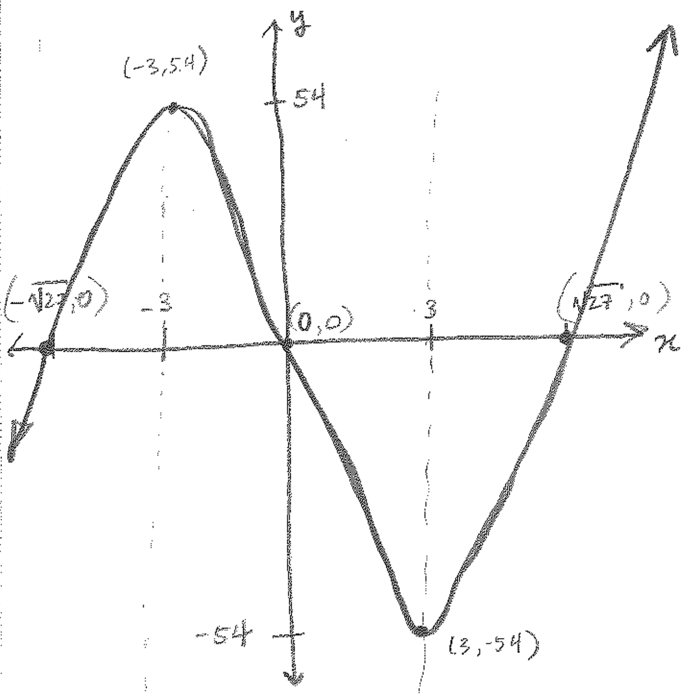
Example. 1, p. 258

$f(x) = x^3 - 27x = x(x^2 - 27) = x(x+\sqrt{27})(x-\sqrt{27})$

$$\left.\begin{aligned} f'(x) &= 3x^2 - 27 \\ &= 3(x^2-9) \\ &= 3(x+3)(x-3) \end{aligned}\right\}$$

So, $f'(x)$ never undefined (no vertical tangents – but we knew that...), and is 0 at $x = -3$ and $x = +3$. Test:





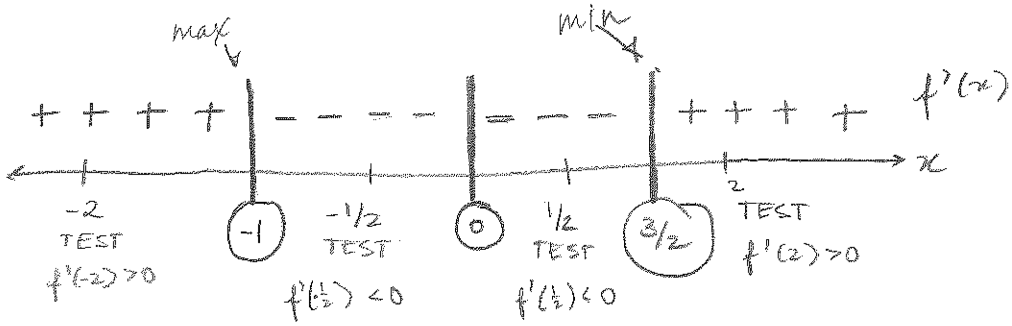
- So f increases on $(-\infty, -3) \cup (3, +\infty)$
  decreases on $(-3, +3)$
- $\therefore$ f has a local max. at $(-3, f(-3)) = (-3, 54)$
  local min. at $(3, f(3)) = (3, -54)$
- f has no global extrema.
- y-intercept: $f(0) = 0$
- x-intercepts: $0 = x(x+\sqrt{27})(x-\sqrt{27})$ at $x = \pm\sqrt{27}$

Example. 2, p. 259 $f(x) = 8x^5 - 5x^4 - 20x^3$, so $f'(x) = 40x^4 - 20x^3 - 60x^2$

$$= 20x^2(2x^2 - x - 3)$$

$$= 20x^2(x+1)(2x-3).$$

$f'(x)$ is always defined, so the only critical points of f occur when $f'(x) = 0$, i.e., when either $x = 0$, $x = -1$ or $x = 3/2$. Let's test the sign of f on the four intervals these 3 points define:



So, f increasing on $(-\infty, -1) \cup (3/2, +\infty)$

decreasing on $(-1, 3/2)$

f has a local max. at $(-1, f(-1)) = (-1, 7)$

a local min. at $(3/2, f(3/2)) = (3/2, -32.0625)$

y-intercept: $f(0) = 0$ so $(0,0)$

x-intercepts: $0 = 8x^5 - 5x^4 - 20x^3 = x^3(8x^2 - 5x - 20)$

$\Rightarrow x^3 = 0$ or $\underbrace{8x^2 - 5x - 20 = 0}_{\Rightarrow x = \frac{5 \pm \sqrt{5^2 - 4(8)(-20)}}{2(8)}} \approx -1.3, 1.9$

so $(0,0)$, $(-1.3, 0)$, $(1.9, 0)$.