Lecture 1

Announcements / Assignments

- READ THE SYLLABUS (it's now posted to myWPI)
- Syllabus quiz on myWPI – complete by Friday 11:59 p.m.
- HW1 posted, due Monday (23 May) 11:59 p.m. – upload to myWPI page
- WW1 posted, due Friday (19 May) 11:59 p.m. – see syllabus for link to WebWork page

Today:

- Antiderivatives (4.8)
- Area / Finite Sums (5.1)
- Sigma notation / Limits of Finite Sums (5.2)
- Definite integrals (5.3) – if time permits

4.8 : Antiderivatives.

From Calc I, we know how to differentiate a function — but some problems require the "inverse" – i.e., we know the derivative, but want to recover the function. (E.g., we know velocity, and want to recover the position).

> **DEF.** A function F is an <u>antiderivative</u> of f on an interval I if, for all x in I, $F'(x) = f(x)$.

<u>Examples.</u>

- $f(x) := 2x$, $I := \mathbb{R}$ ($\mathbb{R}$ is the set of all <u>real</u> numbers)

  $F(x) =$ ____________ is an antiderivative of f on I.

  Check: $F'(x) =$ ____________ .

- $f(x) := \cos(x)$, $I := \mathbb{R}$.

  $F(x) =$ ____________ is an antiderivative of f on I.

  Check:

- $f(x) := e^x$, $I := \mathbb{R}$.

  $F(x) =$ ____________ is an antiderivative of f on I.

<u>Question</u>: Why "<u>an</u> antiderivative", not "<u>the</u> antiderivative" ?

* <u>UNIQUENESS</u> * a very important concept in math !!

... and antiderivatives are not unique, e.g.:

$F_1(x) = x^2 + 1$, $F_2(x) = x^2 + 374$, $F_3(x) = x^2 + (\pi/7)$.

THEOREM p. 281	IF	F is an antiderivative of f on an interval I,
	THEN	the most general antiderivative of f on I is $F(x)+C$, where C is an arbitrary constant.

... so, the "most general" antiderivative is a FAMILY of functions that differ only by a constant — their graphs are vertical translations.

Example 2, p. 282 Find an antiderivative of $f(x) := 3x^2$ that satisfies $F(1) = -1$.

Important: always verify by differentiating!

Examples 3, p.282

Find the general antiderivatives:

(a) $f(x) := x^5$

(b) $g(x) := \frac{1}{\sqrt{x}}$

(c) $h(x) := \sin(2x)$

(d) $l(x) := \cos\left(\frac{x}{2}\right)$

(e) $j(x) := e^{-3x}$

(f) $k(x) := 2^x$

Antiderivative formulas in Table 4.2, p.282 of textbook.

Derivative rules lead naturally to antiderivative rules:

Table 4.3, p.283

If F is the antiderivative of f on I, ~~[illegible]~~
G — " — g on I, and k - constant

Then:

1. $\boxed{k f(x)}$ has antiderivative $\boxed{kF(x) + C}$ on I

2. $\boxed{f(x) + g(x)}$ has antiderivative $\boxed{F(x) + G(x) + C}$ on I.

Example 4, p.283 Find the general antiderivative of

$$f(x) := \frac{3}{\sqrt{x}} + \sin(2x) \quad \text{on } \mathbb{R}.$$

Someday, most of you will continue on to study Ordinary Differential Equations (ODEs). Here is a sneak preview!

DEFS • A <u>differential equation</u> is an eq'n involving an (unknown) function y and its derivative(s) y', y'', etc.

• An <u>initial condition</u> is something like $y(x_0) = y_0$, which specifies the value of y at the point x_0.

Just knowing antiderivatives, you know how to solve some <u>initial value problems</u> of the form:

$$\begin{cases} \dfrac{dy}{dx} = f(x) \\ y(x_0) = y_0 \end{cases}$$

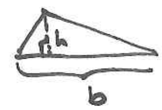
<u>Example.</u> Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = e^{-2t} \\ y(0) = 3. \end{cases}$$

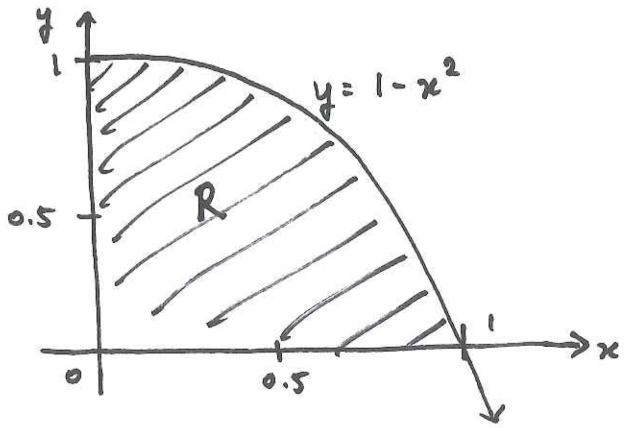
5.1 : Area and Estimating with Finite Sums.

Area.

We know formulas for :

- circle  $A =$ ________
- parallelogram  $A =$ ________
- triangle  $A =$ ________

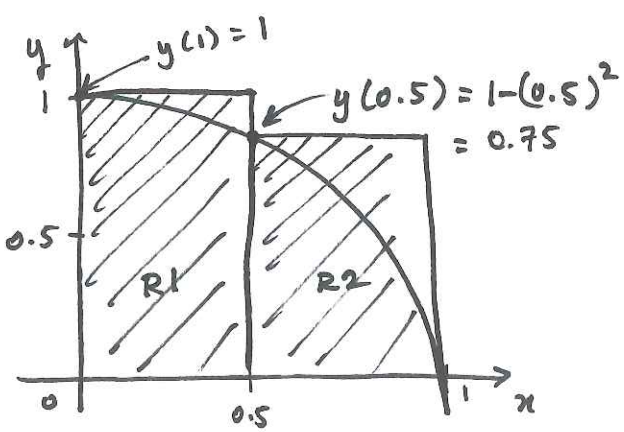
But what if someone asked us to compute the area of R :

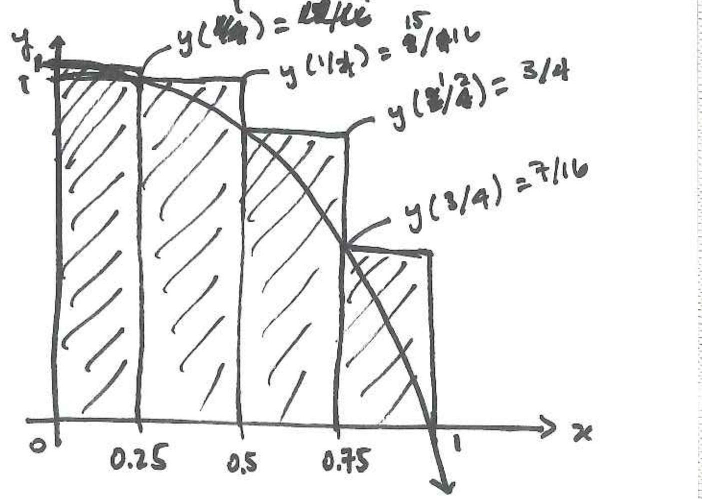


R is above x-axis
below curve $y = 1 - x^2$
between vertical lines $x = 0$, $x = 1$.

We cannot compute using a formula (yet), but can estimate, using rectangles.

Idea : Use rectangles to overshoot the area :





$A(R) \approx A(R1) + A(R2)$

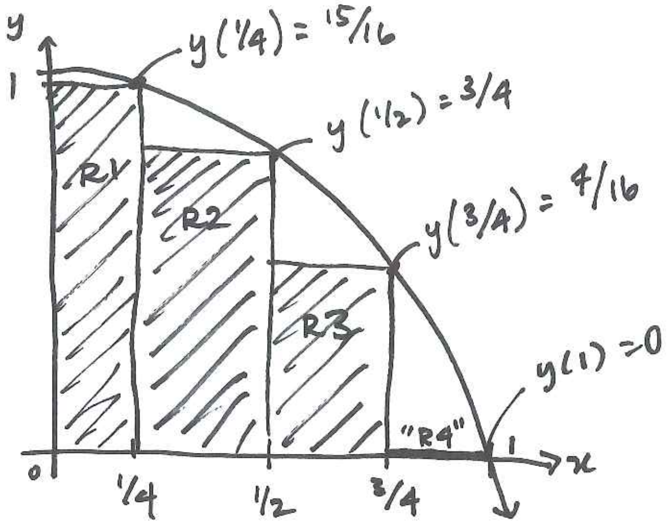
$=$

$A(R) \approx A(R1) + A(R2) + A(R3) + A(R4)$

$=$

"UPPER SUM"

Another idea: Use rectangles to undershoot the area:

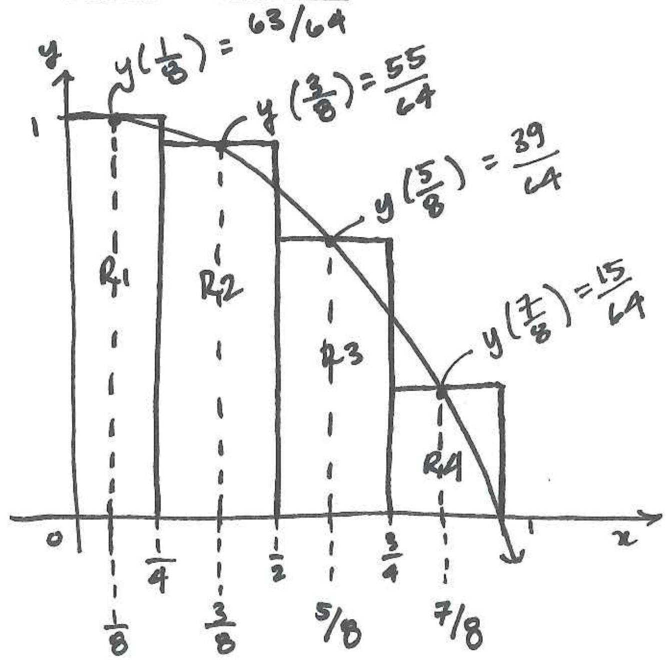


$A(R) \approx A(R1) + A(R2) + A(R3) + A(R4)$

$=$

"LOWER SUM"

... still another idea: Use midpoints to estimate area:



$A(R) \approx A(R1) + A(R2) + A(R3) + A(R4)$

$=$

"MIDPOINT SUM"

So, our estimates using 4 rectangles:

- Upper sum: ____________
- Lower sum: ____________
- Midpoint sum: ____________

Application: Average value (of non-negative, continuous fn.).

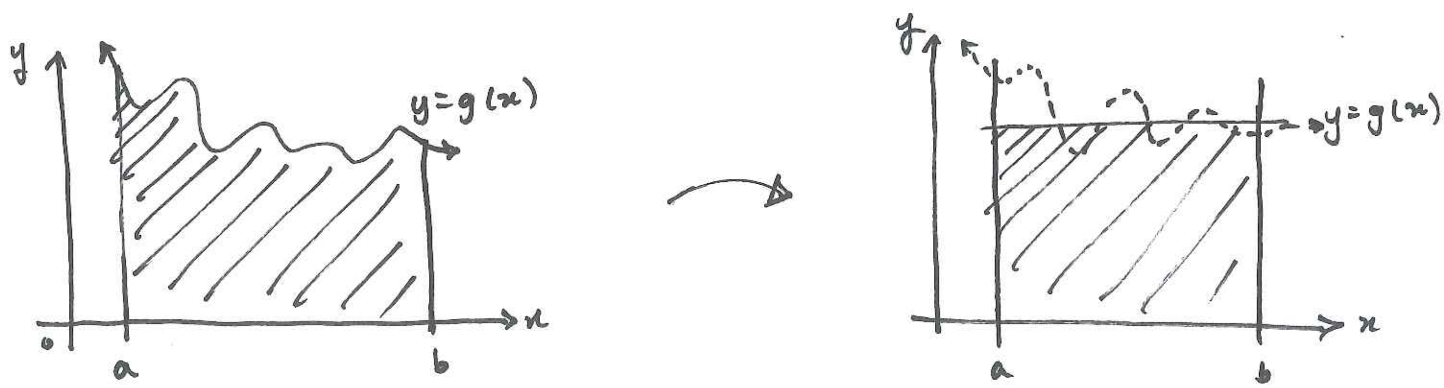
- We know how to compute the average value of a finite collection of numbers — e.g., the average height, age, etc. of students in a class.

$$\text{Ages} = \{19, 22, 18, 20\}$$

$$\text{average} = \frac{\text{sum of all values}}{\text{total \# of samples}} = \frac{19+22+18+20}{4} = \frac{79}{4} = \boxed{19.75}$$

- But what about the average value of a continuous function? — what does it mean when we say the avg. temperature yesterday was 44.6 °F? (Remember, temp. is constantly changing!)

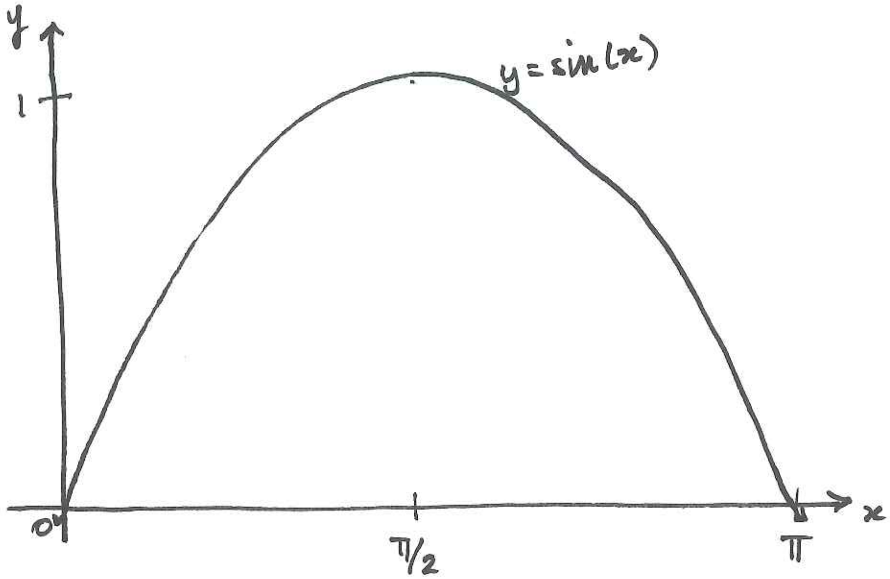
- Idea: think of a function's graph as the level of water in a basin:



* If the graph represented the water level, then the average value is the height of the water, after it settles into the shape of its container *

To compute the average value, we divide the area under the graph by the interval length:

Example. Estimate the avg. val. of $f(x) = \sin(x)$ on $[0, \pi]$.
4, p. 306



Compute area under graph, divide by π.
Let's estimate using the upper sums and eight rectangles:

Avg val $= \dfrac{\text{Area}}{\pi}$, Area $\approx A(1) + A(2) + A(3) + \dots + A(8)$

$$\approx \frac{1}{\pi}\left(\frac{\pi}{8}\left(\sin(0) + \sin(\pi/8) + \sin\left(\frac{2\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{4\pi}{8}\right) + \sin(5\pi/8) + \sin(6\pi/8) + \sin(7\pi/8)\right)\right)$$

$$= \frac{1}{8}\left(\sin(0) + \sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{7\pi}{8}\right)\right)$$

$$= \frac{2.364}{\pi} \approx 0.753.$$

So the avg. val. of $\sin(x)$ over $[0, \pi]$ is APPROXIMATELY 0.753.

5.2: Sigma notation and finite sums / limits thereof.

DEF. Sigma notation gives a compact way of writing a sum with many terms:

index k ends at $k=n$

$$\sum_{k=1}^{n} \underbrace{a_k}_{} := a_1 + a_2 + a_3 + \dots + a_n$$

Greek capital letter sigma ("sum")

a_k is a formula for the k^{th} term

index k starts at $k=1$

Examples.

- $$\sum_{k=2}^{10} k^2 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

- $$\sum_{k=0}^{5} \frac{k}{k+1} = \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} + \frac{5}{5+1}$$
$$= 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$$

- $$\sum_{k=10}^{1000} f(k) = f(10) + f(11) + \dots + f(1000)$$

- $$\sum_{k=\cancel{1}0}^{100\cancel{0}} f(10k) = f(0) + f(10) + f(20) + f(30) + \dots + f(1000)$$

Example. 2, p. 310 Write the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

- start with $k=0$:

$1 = 2\cdot 0 + 1 = 0+1$

$3 = 2\cdot 1 + 1 = 2+1$

$5 = 2\cdot 2 + 1 = 4+1$

$7 = 2\cdot 3 + 1 = 6+1$

so $1+3+5+7+9 = (2(0)+1)+(2(1)+1)+(2(2)+1)+(2(3)+1)$

$$= \sum_{k=0}^{3} 2k+1$$

- start with $k=1$:

$1 = 2\cdot 1 - 1 = 2-1$

$3 = 2\cdot 2 - 1 = 4-1$

$5 = 2\cdot 3 - 1 = 6-1$

$7 = 2\cdot 4 - 1 = 8-1$

So $1+3+5+7+9 = (2(1)-1)+(2(2)-1)+(2(3)-1)+(2(4)-1)$

$= \underline{\hspace{6em}}$.

Proof by induction gives the following rules for finite sums (should be familiar):

Algebra rules for finite sums (p.311)

1. Sum rule: $\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. Constant multiple rule: $\sum_{k=1}^{n} c\cdot a_k = c\sum_{k=1}^{n} a_k$

Examples/Alternate forms

Rule 1: $(a_0+b_0)+(a_1+b_1)+(a_2+b_2)+\dots+(a_n+b_n) =$

$= (a_0+a_1+\dots+a_n) + (b_0+b_1+\dots+b_n)$

Rule 2: $(ca_0)+(ca_1)+(ca_2)+(ca_3)+(ca_4)+\dots+(ca_n) =$

$= c(a_0+a_1+a_2+\dots+a_n)$.

Constant value rule. (p.311)

$$\sum_{k=m}^{n} c = c(n-m+1), \text{ i.e., } \underbrace{c+c+\dots+c}_{n-m+1 \text{ times}} = c(n-m+1).$$

Example 4, p. 311 Show tht. the sum of the 1st n many integers is

$$\sum_{k=1}^{n} n = \frac{n(n+1)}{2}.$$

Reality check: Does it work for $n = 4$?

$$\underbrace{1+2+3+4}_{=} \overset{?}{=} \underbrace{\frac{4(4+1)}{2}}_{=}$$

Proof of general case.

$$\sum_{k=1}^{m} m = 1+2+3+4+\dots+n-2+m-1+m$$

but also $\displaystyle\sum_{k=1}^{m} m = m+(m-1)+(m-2)+(m-3)+\dots+3+2+1$

(just writing backward)

Then $\displaystyle 2\left(\sum_{k=1}^{m} m\right) = \underbrace{(1+m)+(1+m)+\dots+(1+m)}_{n \text{ times}}$ (add)

$$= \sum_{k=1}^{m} 1+m$$

$$= m(1+n)$$

so $\displaystyle\sum_{k=1}^{m} m = \frac{m(1+m)}{2}.$

Formulas for sums of 1st n integers, squares, cubes:

- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

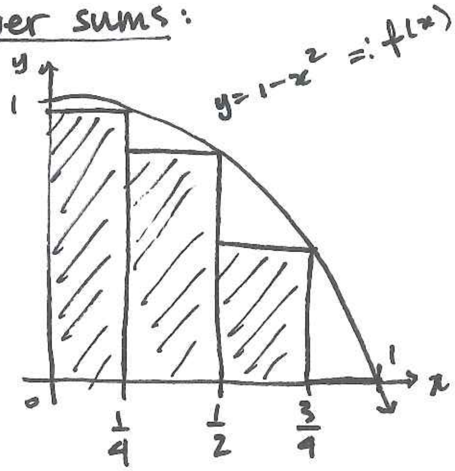
- $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Can prove all by mathematical induction!

<u>Recall</u>: We estimated the area between a curve and the x-axis using upper sums, lower sums, & midpt. sums, using 2, 4, or 8 rectangles ...

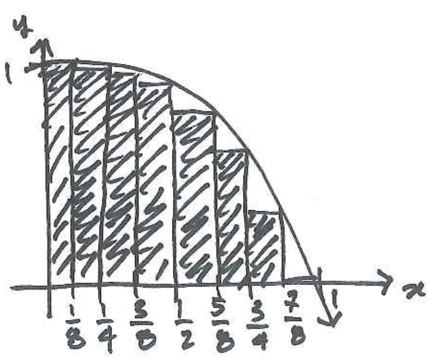
> ... and the approximations got better when we used more rectangles !!

<u>Lower sums</u>:



<u>4 rectangles</u>

$$A \approx A(R1) + A(R2) + A(R3) + A(R4)$$
$$= \frac{1}{4} f(1/4) + \frac{1}{4} f(\tfrac{1}{2}) + \frac{1}{4} f(3/4) + \frac{1}{4} f(1)$$
$$= \frac{1}{4}\left[f(1/4) + f(1/2) + f(3/4) + f(1) \right]$$
$$=$$



<u>8 rect.</u>

$$A \approx A(R1) + A(R2) + A(R3) + \cdots + A(R8)$$
$$= \frac{1}{8} f(1/8) + 1/8\, f(1/4) + \cdots + 1/8\, f(7/8) + 1/8\, f(1)$$
$$= \frac{1}{8}\left[f(1/8) + f(1/4) + f(3/8) + \cdots + f(7/8) + f(1) \right]$$
$$=$$

<u>n many rectangles?</u>

$A \approx$ ______________ .

Recall: For our example, $f(x) = 1 - x^2$.

So for n many rectangles, the lower sum approximation of the area of R is:

$$A \approx \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} 1 - \left(\frac{k}{n}\right)^2.$$ Apply rules!

$$=$$

To get the most accurate approximation, take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \left(\begin{matrix}\text{Lower sum area} \\ \text{w/ } n \text{ many rects}\end{matrix}\right) = \lim_{n\to\infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) =$$

$$= 1 - \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} =$$

So, for our example, the lower sum <u>converges</u> to 2/3.

What about the upper sum?

$$\begin{matrix}\text{Upper sum}\\ \text{for } n \text{ many}\\ \text{rectangles}\end{matrix} = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right), \quad \text{for the interval } [0,1].$$

$$\text{For us, } \begin{matrix}\text{upper}\\ \text{sum}\end{matrix} = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1} 1-\left(\frac{k}{n}\right)^2$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} 1 - \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^2$$

$$= \frac{1}{n}(n-1+1-0) - \frac{1}{n}\left(\frac{1}{n^2}\right)\sum_{k=0}^{n-1} k^2$$

$$= 1 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n^2\right)$$

$$= 1 - \frac{1}{n} - \frac{(n^2+n)(2n+1)}{6}$$

$$= 1 - \frac{1}{n} - \frac{2n^3+3n^2+n}{6n^3}$$

$$= 1 - \frac{1}{n} - \frac{2n^2+3n+1}{6n^2}$$

$$= 1 - \frac{1}{n} - \frac{2n^2}{6n^2} + \frac{3n}{6n^2} + \frac{1}{6n^2}$$

$$= 1 - \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

$$= \frac{2}{3} - \frac{1}{2n} + \frac{1}{6n^2}.$$

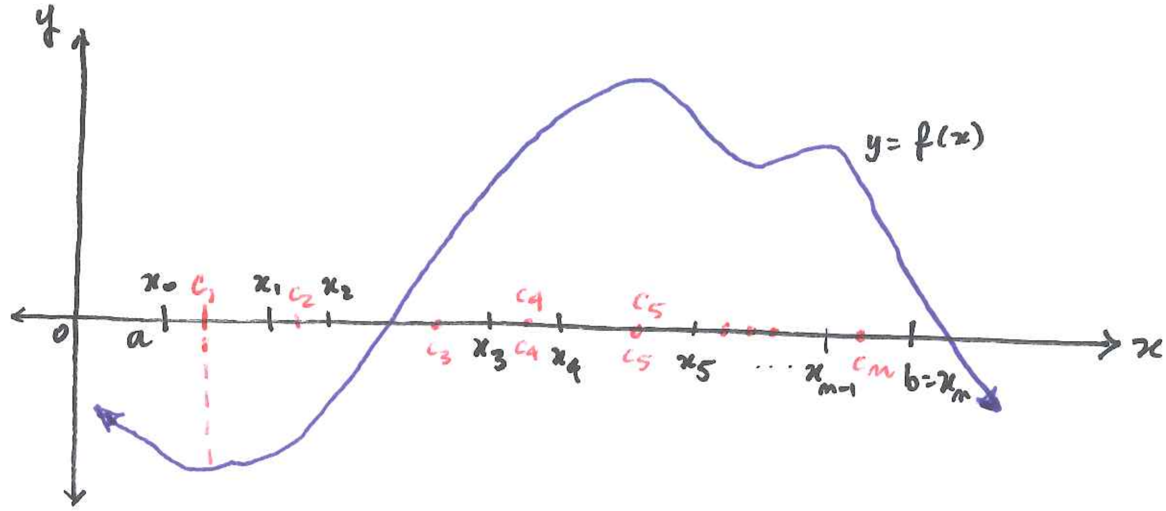
So, limit of the upper sums is :

$$\lim_{n\to\infty}\left(\frac{2}{3}-\frac{1}{2n}+\frac{1}{6n^2}\right) = \frac{2}{3} \quad \text{also.}$$

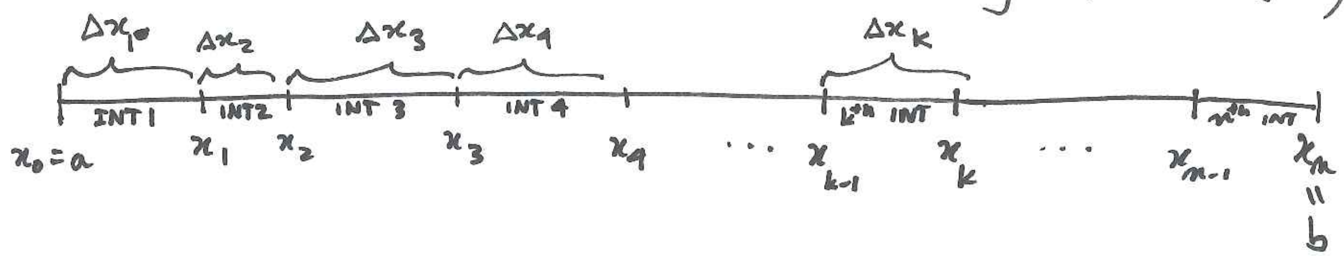
The actual area of R is always between the lower and upper sums, and both the upper and lower sums converge to $\frac{2}{3}$. What does this suggest?

We will (in the next class) make this notion a formal one, but for now, let's introduce the language / methodology of the Riemann sum.

Begin with an arbitrary bounded function f on the closed interval $[a, b]$. (f may have negative values as well as positive ones.)



• Subdivide $[a, b]$ into intervals (not necessarily equal length)



• Call interval lengths $\Delta x_k := x_k - x_{k-1}$.

• Within each (closed) interval, choose a point $c_k \in [x_{k-1}, x_k]$.

"Upper" sums: $c_k = x_{k-1}$ "LEFT ENDPOINTS"

"Lower" sums: $c_k = x_k$ "RIGHT ENDPOINTS"

"Midpoint" sums: $c_k = \frac{x_k + x_{k-1}}{2}$ (mid pt.)

This is where to evaluate the function that gives each "rectangle" height!

- Draw rectangles (optional)

- Area of ~~each~~ k^{th} rectangle is : $\Delta x_k \cdot f(c_k)$

- Sum these areas:

$$S_p := \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

is called ~~the~~ a Riemann sum for f on the interval $[a,b]$.

Can have different Riemann sums depending on our partition "p" of $[a,b]$ into subintervals, and on our choice of the c_k.

Generally, accuracy increases as $n \to \infty$ (for constant-length intervals).