Lecture 10: June 21.

Announcements / Assignments.

- WW 10 due Friday
- HW 5 (final HW) due Monday

Today.

8.3: Trigonometric Integrals

8.4: Trigonometric substitution.

8.3: Trigonometric Integrals.

Basic idea: use trigonometric identities to rewrite integrands so the integrals are easier to evaluate.

Example 1, p. 470 Evaluate $\int \sin^3 x \cos^2 x \, dx$.

FORMULAS: $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$, $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$$\int \sin^3 x \cos^2 x \, dx = \int (1-\cos^2\theta)\cos^2\theta \sin\theta \, d\theta$$

$$= \int (\cos^2\theta - \cos^4\theta)\sin\theta \, d\theta$$

$u := \cos\theta$

$du = -\sin\theta \, d\theta$

$$= -\int u^2 + u^4 \, du$$

$$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \boxed{-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C}$$

Example 2, p. 471 Evaluate $\int \cos^5 x \, dx$.

$\cos^n x = (\cos x)^n$
EXCEPT
$\cos^{-1} x = \arccos x$
unless otherwise stated

$$\int \cos^5 x \, dx = \int (\cos^2 x)^2 \cos x \, dx$$

$$= \int (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

$u := \sin x$
$du = \cos x \, dx$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}$$

Example 3, p. 471 Evaluate $\int \sin^2 x \cos^4 x \, dx$.

$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\int \underbrace{\sin^2 x}\,\underbrace{\cos^4 x}\, dx = \int \underbrace{\left(\frac{1-\cos(2x)}{2}\right)}\underbrace{\left(\frac{1+\cos(2x)}{2}\right)^2} dx$$

$$= \frac{1}{8}\int (1-\cos(2x))(1+2\cos(2x)+\cos^2(2x))\, dx$$

$$= \frac{1}{8}\int 1 + 2\cos(2x) + \cos^2(2x) - \cos(2x) - 2\cos^2(2x) - \cos^3(2x)\, dx$$

$$= \frac{1}{8}\int 1 + \cos(2x) - \underbrace{\cos^2(2x)} - \underbrace{\cos^3(2x)}\, dx$$

$$= \frac{1}{8}\int 1 + \cos(2x) - \overbrace{\left(\frac{1+\cos(2x)}{2}\right)} - \overbrace{(1-\sin^2(2x))\cos(2x)}\, dx$$

$$= \frac{1}{8}\int \frac{1}{2} - \frac{1}{2}\cos(2x)\, dx - \frac{1}{8}\int (1-\sin^2(2x))\cos(2x)\, dx$$

$u := \sin(2x)$

$du = 2\cos(2x)\, dx$

$$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right)\right] - \frac{1}{8}\int \frac{1}{2}(1-u^2)\, du$$

$$= \frac{1}{16}\left[x - \frac{1}{2}\sin(2x)\right] - \frac{1}{16}\left[u - \frac{1}{3}u^3\right] + C$$

$$= \frac{1}{16}\left[x - \frac{1}{2}\sin(2x) - \sin(2x) + \frac{1}{3}\sin^3(2x)\right] + C$$

$$= \boxed{\frac{1}{16}\left[x - \frac{3}{2}\sin(2x) + \frac{1}{3}\sin^3(2x)\right] + C}$$

FORMULA: $\tan^2 x + \sec^2 x = 1$

Example
5, p. 472

Evaluate $\int \tan^4 x \, dx$

$$\int \tan^4 x \, dx = \int \tan^2 x \tan^2 x \, dx = \int \tan^2 x (1 - \sec^2 x) \, dx$$

$$= \int \tan^2 x \, dx - \int \tan^2 x \sec^2 x \, dx$$

$$= \int 1 \, dx - \int \sec^2 x \, dx - \int \tan^2 x \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int 1 \, dx - \int \sec^2 x \, dx - \int u^2 \, du$$

$$= x - \tan x - \frac{1}{3} u^3 + C$$

$$= \boxed{x - \tan x - \frac{1}{3} \tan^3 x + C}$$

$\sec^2 x = \tan^2 x + 1$

Example 6, p.473 Evaluate $\int \sec^3 x \, dx$.

$$\cancel{\int \sec^3 x \, dx = \int (\tan^2 x + 1) \sec x \, dx}$$

$$\cancel{= \int \tan^2 x \sec x + \sec x \, dx}$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$u := \sec x \qquad v = \tan x$

$du = \sec x \tan x \, dx \qquad dv := \sec^2 x \, dx$

$\int u \, dv = uv - \int v \, du$

$$\frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{d}{dx}\left[(\cos x)^{-1}\right]$$

$$= -1 (\cos x)^{-2}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

So we get the algebraic eq'n:

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx$$

$$= \boxed{\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C}$$

$\sec^2 x = \tan^2 x + 1$

Example
7, p.473

$$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \sec^2 x \, dy$$

$$= \int \underbrace{\tan^4 x (\tan^2 x + 1)} \underbrace{\sec^2 x \, dx}$$

$u := \tan x$
$du = \sec^2 x \, dx$

$$= \int u^4 (u^2 + 1) \, du$$

$$= \int u^6 + u^4 \, du$$

$$= \frac{1}{7} u^7 + \frac{1}{5} u^5 + C$$

$$= \boxed{\frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x + C}$$

FORMULAS.

$$\sin(m\theta)\sin(n\theta) = \frac{1}{2}\left[\cos(m-n)\theta - \cos(m+n)\theta\right]$$

$$\sin(m\theta)\cos(n\theta) = \frac{1}{2}\left[\sin(m-n)\theta + \sin(m+n)\theta\right]$$

$$\cos(m\theta)\cos(n\theta) = \frac{1}{2}\left[\cos(m-n)\theta + \cos(m+n)\theta\right]$$

Example. 8, p. 474

Evaluate $\int \sin(3x)\cos(5x)\,dx$.

$$\int \sin(3x)\cos(5x)\,dx = \int \frac{1}{2}\left[\sin((3-5)x) + \sin((3+5)x)\right] dx$$

$\sin(-\theta) = -\sin(\theta), \forall\theta$

i.e., sin is an "odd fn."

$$= \frac{1}{2}\int \sin(-2x) + \sin(8x)\,dx$$

$$= \frac{1}{2}\int \sin(8x) - \sin(2x)\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos(8x) + \frac{1}{2}\cos(2x)\right] + C$$

$$\boxed{= \frac{1}{4}\left[\cos(2x) - \frac{1}{4}\cos(8x)\right] + C}$$

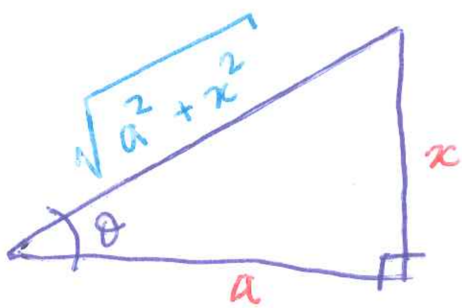
8.4: Trigonometric substitution.

Trig. substitution gives us a way of evaluating integrals involving the forms

$$\sqrt{a^2+x^2}, \quad \sqrt{a^2-x^2}, \quad \sqrt{x^2-a^2}.$$

The substitutions come from the following scenarios: " $\sin\theta = \frac{O}{H}$, $\cos\theta = \frac{A}{H}$, $\tan\theta = \frac{O}{A}$ ", "S $\frac{O}{H}$ C $\frac{A}{H}$ T $\frac{O}{A}$"

1



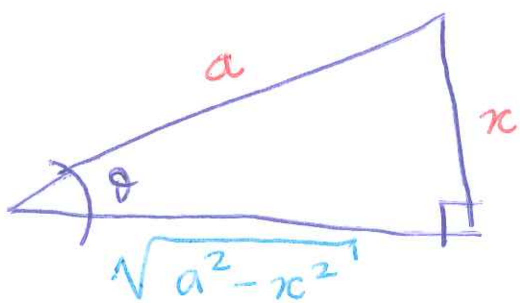
$\tan\theta = x/a$

$x = a\tan\theta$

$\cos\theta = \dfrac{a}{\sqrt{a^2+x^2}}$

$\sqrt{a^2+x^2} = a|\sec\theta|$

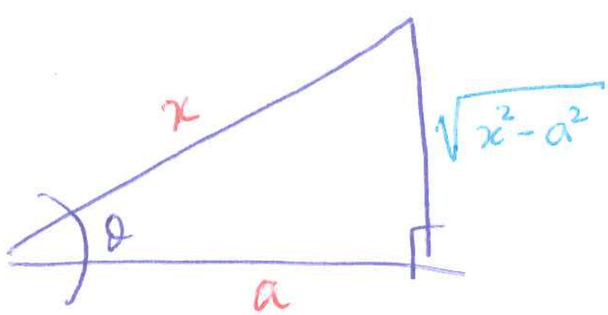
2



$\sin\theta = \dfrac{x}{a} \Rightarrow x = a\sin\theta$

$\cos\theta = \dfrac{\sqrt{a^2-x^2}}{a} \Rightarrow \sqrt{a^2-x^2} = a|\cos\theta|$

3



$\cos\theta = \dfrac{a}{x} \Rightarrow x = a\cos\theta$

$\tan\theta = \dfrac{\sqrt{x^2-a^2}}{a} \Rightarrow \sqrt{x^2-a^2} = a\tan\theta$

To use trig substitution ...

1. Write down the substitution for x

   Calculate the differential dx

   Specify a range for θ-values } p. 476 for ranges of the trig fns / domains of arc-trigfns.

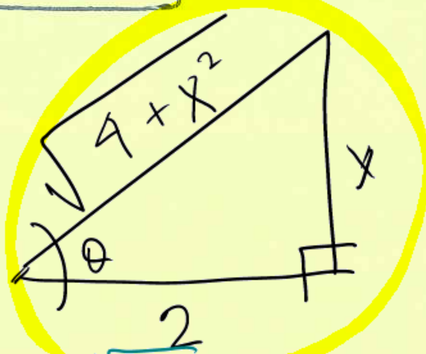
2. Substitute the expression & differential and simplify algebraically

3. Integrate (keep in mind the θ-range)

4. Draw a reference triangle, reverse the substitution.

Example 1, p. 476

Evaluate $\int \frac{1}{\sqrt{4+x^2}}\,dx$.



$\tan\theta = \frac{x}{2} \Rightarrow x = 2\tan\theta$ $\Rightarrow \theta = \arctan(\frac{1}{2}x)$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow dx = 2\sec^2\theta\, d\theta$

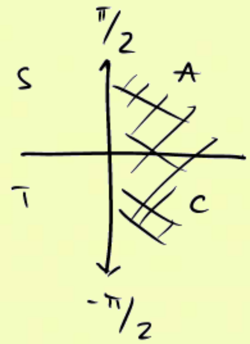
$\cos\theta = \frac{2}{\sqrt{4+x^2}} \Rightarrow \sqrt{4+x^2} = \frac{2}{|\cos\theta|} = 2|\sec\theta|$

$\sec\theta = \frac{\sqrt{4+x^2}}{2}$

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{|\cos\theta|}{2} \cdot 2\sec^2\theta\, d\theta$$

$= \int (4+x^2)^{-1/2}\,dx$

$$= \int \frac{|\cos\theta|}{\cos^2\theta}\,d\theta$$



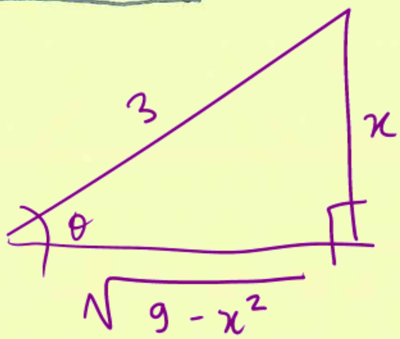
$$= \int \frac{1}{\cos\theta}\,d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\boxed{= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C}$$

Example 3, p. 477 Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$.



$\sin\theta = \dfrac{x}{3} \Rightarrow x = 3\sin\theta \Rightarrow \theta = \arcsin\left(\dfrac{x}{3}\right)$

$\Rightarrow dx = 3\cos\theta\, d\theta \;\Big|\; \Rightarrow \theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\cos\theta = \dfrac{\sqrt{9-x^2}}{3} \Rightarrow \sqrt{9-x^2} = 3\,|\cos\theta|$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^2}{3\,|\cos\theta|}\,3\cos\theta\,d\theta$$

because $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta \geq 0$, i.e., $|\cos\theta| = \cos\theta$

$$= \int 9\sin^2\theta\,d\theta$$

Recall
$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

$$= \int \frac{9}{2}\left(1-\cos(2\theta)\right)d\theta$$

$$= \frac{9}{2}\Big(\underbrace{\theta}_{\theta = \arcsin\left(\frac{x}{3}\right)} - \frac{1}{2}\sin(2\theta)\Big) + C$$

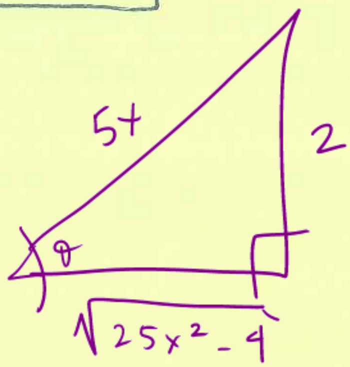
Recall:
$\frac{1}{2}\sin(2\theta) = \sin\theta\cos\theta$

$$= \frac{9}{2}\Big(\theta - \underbrace{\sin\theta}_{x/3}\,\underbrace{\cos\theta}_{\frac{\sqrt{9-x^2}}{3}}\Big) + C$$

$$= \boxed{\frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x}{3} - \frac{\sqrt{9-x^2}}{3}\right) + C}$$

8.4, ct'd.

Example 4, p. 478 Evaluate $\int \frac{1}{\sqrt{25x^2-4}}\,dx$, $x > \frac{2}{5}$.



$\sin\theta = \frac{2}{5x} \Rightarrow x = \frac{2}{5\sin\theta} = \frac{2}{5}\csc\theta$

$\Rightarrow dx = \frac{2}{5}\frac{d}{d\theta}\left[\csc\theta\right]d\theta$

$= \frac{2}{5}\frac{d}{dx}\left[(\sin\theta)^{-1}\right]d\theta$

$= \frac{2}{5}(-1)(\sin\theta)^{-2}\cos\theta\,d\theta$

$= -\frac{2}{5}\cot\theta\csc\theta\,d\theta$

$\Rightarrow \theta = \arcsin\left(\frac{2}{5x}\right),\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\csc\theta = \frac{1}{\sin\theta} = \frac{5x}{2}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{25x^2-4}}{2}$

$\tan\theta = \frac{2}{\sqrt{25x^2-4}} \Rightarrow \frac{1}{\sqrt{25x^2-4}} = \frac{1}{2}\tan\theta$

$$\int \frac{1}{\sqrt{25x^2-4}}\,dx = \int \frac{1}{2}\tan\theta\left(-\frac{2}{5}\cot\theta\csc\theta\right)d\theta$$

$$= -\frac{1}{5}\int \csc\theta\,d\theta$$

$$= -\frac{1}{5}\left(-\ln\left|\csc\theta + \cot\theta\right|\right) + C$$

$$= \frac{1}{5}\ln\left|\csc\theta + \cot\theta\right| + C$$

$$= \boxed{\frac{1}{5}\ln\left|\frac{5x + \sqrt{25x^2-4}}{2}\right| + C}$$