Lecture 11 : June 23.

Announcements/ Assignments:

- WW 10 due Friday
- WW 11 due Monday
- HW 5 due Monday (I will accept on Tuesday or Thursday in hard-copy only)

- Final Exam 1 week from today!
  - Online students: if you will use a proctoring service, have them contact me!
  - If you want to take the exam at the beginning of A-term, contact me (email: EMKILEY@WPI.EDU)
  - Start making 8.5×11" note sheet!

Today:

- 8.5: Integration using Partial Fractions
- 8.7: Numerical integration (as time permits)

8.5: Integration by Partial Fractions.

Recall: We can add fractions by putting them over a common denominator:

Example.

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)+3(x+1)}{(x+1)(x-3)}$$

$$= \frac{2x-6+3x+3}{x^2-2x-3}$$

$$= \frac{5x-3}{x^2-2x-3}$$

Question. If we are just given one fraction whose denominator factors, how can we "reverse" this process? e.g.,

$$\frac{5x-3}{x^2-2x-3} = \frac{?}{x-3} + \frac{?}{x+1}$$

The "reversal" is a technique called PARTIAL FRACTION DECOMPOSITION.

METHOD OF PARTIAL FRACTION DECOMPOSITION.

1. Make sure fraction is proper – i.e., degree of numerator less than degree of denominator [$\frac{3x}{x^3+27}$ is proper, but $\frac{x^7+1}{x^5-8}$ is not.]. If fraction is improper, then do polynomial division instead.

2. Factor denominator:

$$\frac{f(x)}{g(x)} = \frac{f(x)}{\underbrace{(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots}_{\text{linear factors}}\ \underbrace{(x^2+p_1x+q_1)^{m_1}\cdots}_{\text{quadratic factors}}\ \text{etc.}}$$

3. Write the partial fraction sum:

   3a. For the linear factor $(x-r)^m$, include terms:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

   3b. For the quadratic factor $(x^2+px+q)^m$, include

$$\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_mx+C_m}{(x^2+px+q)^m}$$

PARTIAL FRACTION DECOMPOSITION, CT'D.

(3) ct'd. Do this for all factors.

(4) Set $\dfrac{f(x)}{g(x)}$ equal to the partial fraction sum.

(5) Multiply both sides of (4) by $g(x)$

(6) Solve for the unknown coefficients A_i, B_i, C_i, etc. Either:

(6a) Equate coefficients of corresponding powers of x

or

(6b) Substitute several "convenient" x-values.

(7) Check your work!

<u>Example 1, p. 481</u> Use partial fractions to evaluate

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx.$$

Set

$$\cancel{(\cdot)(\cdot)(\cdot)}\left(\frac{x^2+4x+1}{\cancel{(x-1)(x+1)(x+3)}}\right) = \left(\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}\right)(\cdot)(\cdot)(\cdot)$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x+1)(x-1)$$

One way:

<u>let $x=1$:</u>

$$1^2+4(1)+1 = A(1+1)(1+3) + \cancel{B(0)(4)} \cancel{+C(2)(0)}$$

$$1+4+1 = A(2)(4)$$

$$6 = 8A \Rightarrow A = \frac{6}{8} = \frac{3}{4}.$$

let $x = -1$:

$$(-1)^2 + 4(-1) + 1 = \cancel{A\underbrace{(-1+1)}_{0}(-1+3)} + B(-1-1)(-1+3) + \cancel{C\underbrace{(-1+1)}_{0}(-1-1)}$$

$$1 - 4 + 1 = B(-2)(2)$$

$$-2 = -4B \Rightarrow B = \frac{-2}{-4} = \frac{1}{2}.$$

let $x = -3$:

$$(-3)^2 + 4(-3) + 1 = \cancel{A(-3+1)\underbrace{(-3+3)}_{0}} + \cancel{B(-3-1)\underbrace{(-3+3)}_{0}} + C(-3+1)(-3-1)$$

$$9 - 12 + 1 = C(-2)(-4)$$

$$-2 = 8C \Rightarrow C = \frac{-2}{8} = \frac{-1}{4}.$$

$$\boxed{\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{3}{4(x-1)} + \frac{1}{2(x+1)} - \frac{1}{4(x+3)}}$$

$$= \frac{\frac{3}{4}(x+1)(x+3) + \frac{1}{2}(x-1)(x+3) - \frac{1}{4}(x+1)(x-1)}{(x-1)(x+1)(x+3)}$$

So the integral is :

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \frac{3}{4(x-1)} + \frac{1}{2(x+1)} - \frac{1}{4(x+3)}\,dx$$

$$= \frac{3}{4}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{4}\int \frac{1}{x+3}\,dx$$

$u := x-1$ $\quad$ $u := x+1$ $\quad$ $u := x+3$

$du = dx$ $\quad$ $du = dx$ $\quad$ $du = dx$

$$= \frac{3}{4}\int \frac{1}{u}\,du + \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{3}{4}\ln|u| + \frac{1}{2}\ln|u| - \frac{1}{4}\ln|u|$$

$$= \boxed{\frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C}$$

Example 2, p. 482

Use partial fractions to evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx$$

let

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

(step 3a in table)

(DO NOT SAY:

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{x+2}$$

$$\frac{6x+7}{(x+2)^2} = \frac{A+B}{x+2} \quad ??)$$

$$6x + 7 = A(x+2) + B = Ax + 2A+B$$

$$6\cancel{x} = A\cancel{x} \Rightarrow A = 6$$

$$7 = 2A + B = 2(6) + B = 12 + B$$

$$\Rightarrow 7 - 12 = B \Rightarrow B = -5$$

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}$$

Check:

$$\frac{6x+7}{(x+2)^2} \stackrel{?}{=} \frac{6(x+2)-5}{(x+2)^2} = \frac{6x+12-5}{(x+2)^2}$$

$$= \frac{6x+7}{(x+2)^2}$$

So:

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \frac{6}{x+2} - \frac{5}{(x+2)^2}\,dx$$

$$= 6\int \frac{1}{x+2}\,dx - 5\int \frac{1}{(x+2)^2}\,dx$$

$$= 6\int \frac{1}{x+2}\,dx - 5\int (x+2)^{-2}\,dx$$

$$= 6\ln|x+2| - \frac{5}{-1}(x+2)^{-1} + C$$

$$= \boxed{6\ln|x+2| + \frac{5}{x+2} + C}$$

Example 3, p. 482 Evaluate $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$.

$$\begin{array}{r|l} & 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & -(2x^3-4x^2-6x) \\ \hline & \qquad\qquad 5x-3 \end{array}$$

So,

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3} = 2x + \frac{5x-3}{(x-3)(x+1)}$$

Let

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$5x-3 = A(x+1) + B(x-3)$$

If $x=-1$: $5(-1)-3 = B(-1-3) \iff -8 = -4B \Rightarrow B=2$

If $x=3$: $5(3)-3 = A(3+1) \iff 12 = 4A \Rightarrow A=3$

So $\frac{5x-3}{x^2-2x-3} = \frac{3}{x-3} + \frac{2}{x+1}$

check: $3(x+1)+2(x-3) =$
$= 5x-3$ ✓

So

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x + \frac{3}{x-3} + \frac{2}{x-1}\,dx$$

$$= \boxed{x^2 + 3\ln|x-3| + 2\ln|x-1| + C}$$

$\boxed{\text{Example 4, p. 483}}$ Evaluate $\displaystyle\int \frac{-2x+4}{\underbrace{(x^2+1)^1}_{\text{"irreducible quadratic"}}(x-1)^2}\,dx.$

Let

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$-2x+4 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$-2x+4 = A[x^3-x^2+x-1] + B[x^2+1] + C[x^3-2x^2+x] + D[x^2-2x+1]$$

$$\boxed{0x^3} + 0x^2 - 2x + 4 = \boxed{x^3[A+C]} + \boxed{x^2[-A+B-2C+D]} + \boxed{x[A+C-2D]} + \boxed{[-A+B+D]}$$

1 $\quad 0 = A + C$ $\qquad$ 2 $\quad 0 = -A + B - 2C + D$

3 $\quad -2 = A + C - 2D$ $\qquad$ 4 $\quad 4 = -A + B + D$

(Eq. 2) − (Eq. 4):

$$\begin{aligned} 0 &= \cancel{-A} + \cancel{B} - 2C + \cancel{D} \\ -(4 &= \cancel{-A} + \cancel{B} \qquad\quad + \cancel{D}) \\ -4 &= -2C \quad\Rightarrow\quad C = 2. \end{aligned}$$

$(\text{Eq.}1)$: $\quad 0 = A + C \Rightarrow A = -C \Rightarrow A = -2.$

$(\text{Eq.}3)$: $\quad -2 = A + C - 2D \Rightarrow -2 = -2D \Rightarrow D = 1$

$(\text{Eq.}4)$: $\quad 4 = -A + B + D \Rightarrow 4 = -(-2) + B + 1$

$\Rightarrow 4 = 3 + B \Rightarrow B = 1.$

So
$$\frac{-2x+4}{(x^2+1)(x-1)^2} \overset{?}{=} \frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}$$

$$= \frac{-2(x-1)(x^2+1) + x^2+1 + (2x+1)(x-1)^2}{(x-1)^2(x^2+1)}$$

$$= \frac{-2(x^3+x-x^2-1) + x^2+1 + (2x+1)(x^2-2x-1)}{(x-1)^2(x^2+1)}$$

$$= \frac{\cancel{-2x^3} - 2x + \cancel{2x^2} + 2 + \cancel{x^2} + 1 + \cancel{2x^3} - \cancel{4x^2} - 2x + \cancel{x^2} - 2x - 1}{(x-1)^2(x^2+1)}$$

$$= \frac{-2x - 2x - 2x + 2 + 1 - 1}{(x-1)^2(x^2+1)}$$

$$-2x+4 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

let $x=1$

$$-2(1)+4 = \cancel{A(0)(2)} + B(1^2+1) + \cancel{(C+D)(0)^2}$$

$$2 = 2B \Rightarrow B=1$$

$$-2x+4 = A(x-1)(x^2+1) + x^2+1 + (Cx+D)(x-1)^2$$

$$0x^3 \boxed{-x^2-2x+3} = A[x^3+x-x^2-1] + C[x^3-2x^2+x] + D[x^2-2x+1]$$

$$= x^3[A+C] + x^2[-A-2C+D] + x[A+C-2D] + [-A+D]$$

(1) $0 = A+C$ (2) $-1 = -A-2C+D$

(3) $A+C-2D = -2$ (4) $3 = D-A$

(4) ; (2) $\Rightarrow -1 = 3-2C \Rightarrow 2C = 3+1 \Rightarrow C=2$

(1) ; $C=2 \Rightarrow A=-2$; (4) ; $A=-2 \Rightarrow 3 = D+2 \Rightarrow D=1$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}$$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}\,dx$$

$$= -2\int \frac{1}{x-1}\,dx + \int (x-1)^{-2}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= -2\ln|x-1| - (x-1)^{-1} + \int \frac{2x}{x^2+1}\,dx + \int \underbrace{\frac{1}{x^2+1}}_{\text{tab. 8.1: } = \arctan(x)}\,dx$$

$u := x^2+1$

$$\underbrace{du = 2x\,dx}$$

$$\int \frac{1}{u}\,du = \ln|x^2+1| + c$$

$$= \ln(x^2+1) + c$$

$$= \boxed{-2\ln|x-1| - \frac{1}{x-1} + \ln(x^2+1) + \arctan(x) + C}$$

Example. 5, p. 484 Evaluate $\int \frac{1}{x(x^2+1)^2}\,dx$.

Let

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+c)x(x^2+1) + (Dx+E)x$$

$$= A[x^4+2x^2+1] + B[x^4+x^2] + C[x^3+x] + D[x^2] + E[x]$$

$$0x^4 + 0x^3 + 0x^2 + 0x + 1 = x^4[A+B] + x^3[C] + x^2[2A+B+D] + x[C+E] + A$$

(1) $0 = A+B$ (2) $0 = C$ (3) $0 = 2A+B+D$

(4) $0 = C+E$ (5) $A = 1$

① & ⑤ imply $B = -A = -1$, so $B = -1$

③ & $B=-1$ & ⑤ imply $D = -B - 2A$

$= -(-1) - 2(1) = 1 - 2 = -1$

so $D = -1$

④ & ② imply $E = -C = -(0) = 0$, so $E = 0$

In summary: $A = 1$, $B = -1$, $C = 0$, $D = -1$, $E = 0$

$$\frac{1}{x(x^2+1)^2} \overset{\checkmark}{=} \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}$$

$$= \frac{(x^2+1)^2 - x(x^2+1)x - x(x)}{x(x^2+1)^2}$$

$$= \frac{\cancel{x^4} + \cancel{2x^2} + 1 - \cancel{x^4} \cancel{- x^2} \cancel{- x^2}}{x(x^2+1)^2}$$

$$= \frac{1}{x(x^2+1)^2}$$

So:

$$\int \frac{1}{x(x^2+1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

$u := x^2+1$

$du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\int u^{-2}\,du$$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + C$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + C$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C}$$

8.5, ct'd,

Example Decompose $\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$

6, p. 484

into partial fractions.

NOTE: p. 484,

"cover-up method"

we've already done –

it is optional (you can

do next 3 examples

without it)

8.5 ctd

Example 7, p. 485

Evaluate $$\int \frac{x+4}{x^3+3x^2-10x}\,dx.$$

$\boxed{\text{Example. } 8, \text{ p. } 486}$

Decompose $\dfrac{x-1}{(x+1)^3}$ into partial fractions.

$\boxed{\text{Example. } 9, \text{ p. } 487}$ Decompose $\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$ into partial fractions.