Lecture 12 (Final Lecture): June 28.

Announcements / Assignments.

- WebWork 12 due before 11:59 p.m. Friday — but suggested to complete before exam Thursday
- Homework 5 due at final exam
- Final exam Thursday 6-8 p.m. in OH 109
- Online / off-campus students:

  e-mail me with your intentions!

  (a) come to campus Thursday

  (b) take exam remotely with proctor Thursday 6-8 pm EDT

  (c) delay exam until start of A-term, take on campus at WPI

  You __must__ let me know which option you prefer!

- Look at your grades on myWPI & WebWork, and compare to syllabus to see where you stand

TODAY:
- 8.7
- Exam review

8.7: Numerical Integration.

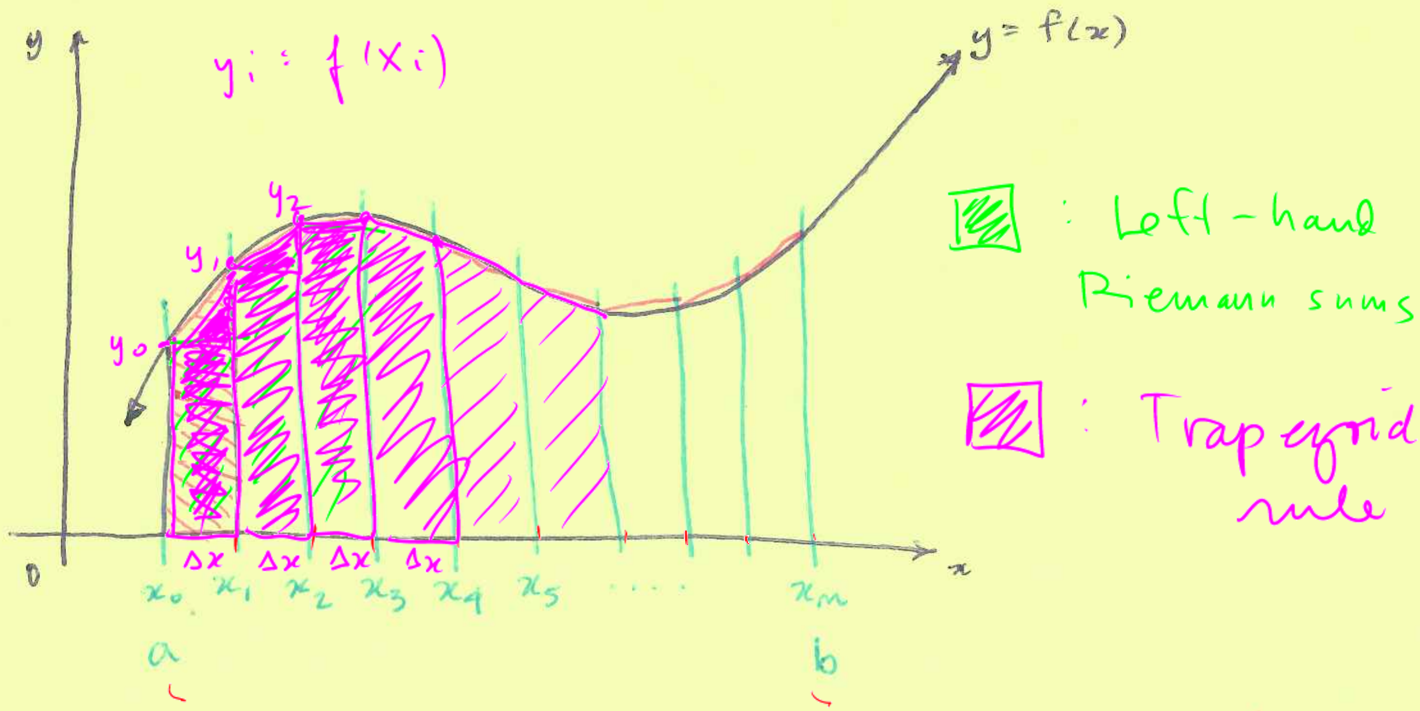
Some functions, e.g., $\sin(x^2)$, $\frac{1}{\ln(x)}$, $\sqrt{1+x^4}$, have no antiderivative expressible using elementary functions. How to compute definite integrals?

Hint: You already know one good way of approximating:

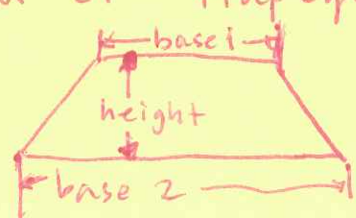
Riemann Sums

There are two other ways we will learn.

1. Trapezoidal Rule.



Area of trapezoid : height$\left(\frac{\text{base 1} + \text{base 2}}{2}\right)$



Trapezoidal rule:

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} (\Delta x)\left(\frac{y_i + y_{i+1}}{2}\right)$$

$$= \frac{\Delta x}{2} \sum_{i=0}^{n-1} y_i + y_{i+1}$$

Formula for Trapezoid Rule

$$= \frac{\Delta x}{2}\Big[(y_0 + \underbrace{y_1) + (y_1}+\underbrace{y_2) + (y_2}+\underbrace{y_3) + \cdots}$$
$$\cdots + (y_{n-1} + y_n)\Big]$$

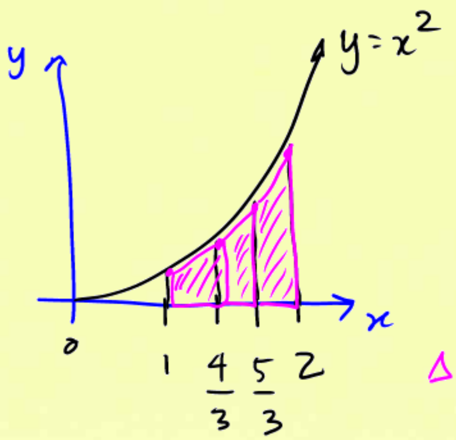
$$\boxed{\int_a^b f(x)\,dx = \frac{\Delta x}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n\right]}$$

where the y_i are the function values $f(x_i)$ and where Δx is the (uniform) node spacing.

$\boxed{\text{Example 1, p.496}}$ Use Trapezoidal rule w/ $n=4$ to estimate $\int_1^2 x^2\,dx$. compare w/ exact value.

$f(x) := x^2$

$$\int_1^2 x^2\,dx \approx \frac{\Delta x}{2}\left[y_0 + 2y_1 + 2y_2 + y_3\right]$$



$$= \frac{1/3}{2}\left[f(1) + 2f(4/3) + 2f(5/3) + f(2)\right]$$

$$= \frac{1}{6}\left[1^2 + 2\left(\frac{4}{3}\right)^2 + 2\left(\frac{5}{3}\right)^2 + 2^2\right]$$

$\Delta x = \frac{1}{3}$

$$= \frac{1}{6}\left[1 + \frac{32}{9} + \frac{50}{9} + 4\right]$$

$$= \frac{1}{6}\left[\frac{9 + 32 + 50 + 36}{9}\right]$$

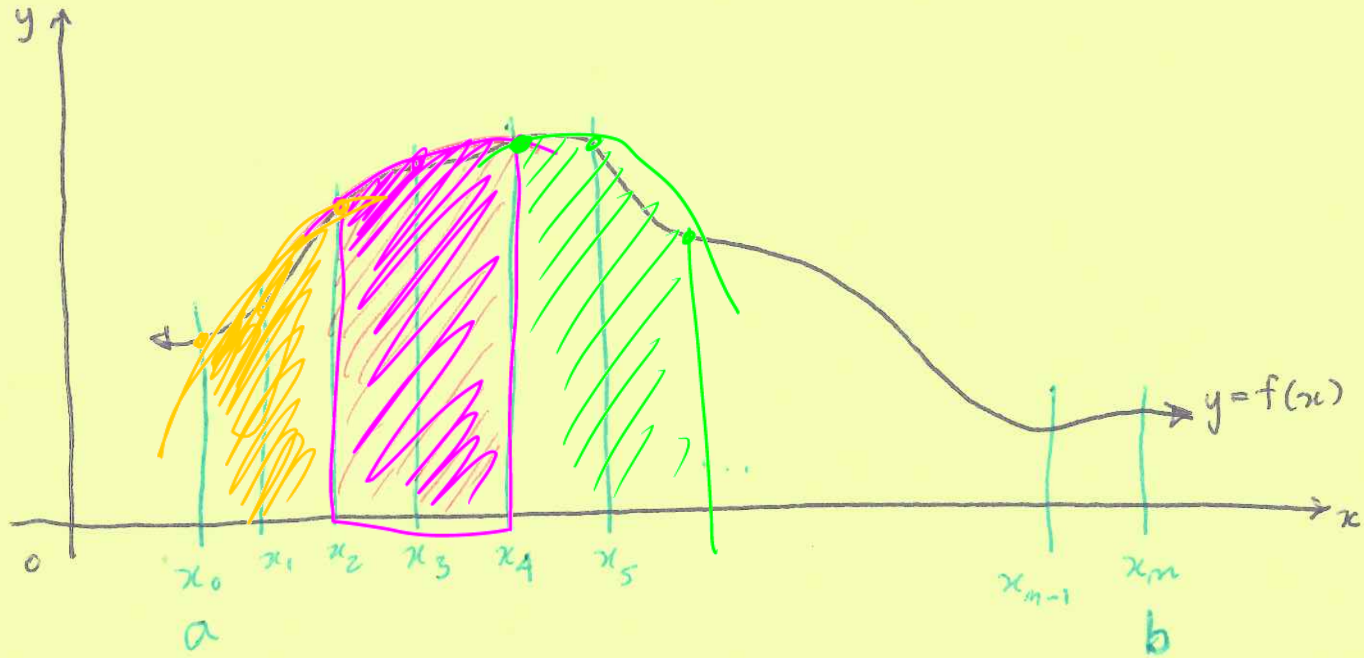
$$= \frac{127}{54} \approx 2.35$$

Exact: $\int_1^2 x^2\,dx = \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}\left(2^3 - 1^3\right) = \frac{7}{3} \approx 2.\overline{3}$

Difference is $\frac{127}{54} - \frac{7}{3} = \frac{127 - 7\cdot 18}{54} = \frac{1}{54}$

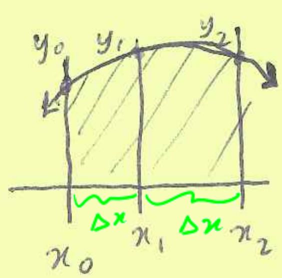
8.7, Numerical Integration, ct'd.

Second method: Simpson's Rule.



IDEA: Use parabolas instead of trapezoids.

Fact: The area beneath the segment of the parabola that connects (x_0, y_0), (x_1, y_1) and (x_2, y_2), between $x_0 < x < x_2$, is



$$A = \frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right)$$

where Δx is the (uniform) node spacing.

Use this for each parabola:

Simpson's Rule.

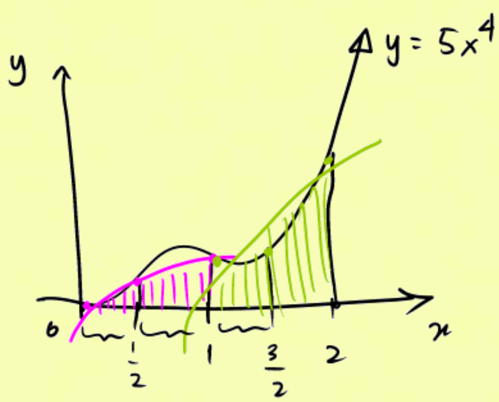
$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big(y_0 + \underbrace{4y_1 + 2y_2} + \underbrace{4y_3 + 2y_4} + \cdots \underbrace{\quad}$$
$$\underbrace{+ 2y_{n-2}} + 4y_{n-1} + y_n\Big)$$

where the y_i are the values $f(x_i)$, and Δx is the (uniform) node spacing.

★ NOTE: If nodes are $x_0, x_1, \dots, x_n$, then total # of nodes is $n+1$, and we need $(n+1)-3 = n-2$ to be divisible by 2

Example. Use Simpson's Rule with $n=4$ to estimate

$$\int_0^2 5x^4\,dx. \quad \text{compare to reality.}$$



$\Delta x = 1/2$

$$\int_0^2 5x^4\,dx = \frac{\Delta x}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + f(2)\right]$$

$$= \frac{1/2}{3}\left[5(0)^4 + 4\cdot 5(1/2)^4 + 2\cdot 5(1)^4 + 4\cdot 5(3/2)^4 + 5(2)^4\right]$$

$$= \frac{5}{6}\left[0 + 4\cdot\left(\frac{1}{16}\right) + 2(1) + 4\left(\frac{81}{16}\right) + 16\right]$$

$$= \frac{5}{6}\left[\frac{4 + 32 + 4\cdot 81 + 16^2}{16}\right]$$

$$= \frac{3080}{96}$$

In reality, $\int_0^2 5x^4\,dx = x^5\Big|_0^2 = 2^5 - 0^5 = 32$

The error was $\left|\frac{3080}{96} - 32\right| = \left|\frac{3080-3072}{96}\right|$

$$= \frac{8}{96} = \frac{1}{12}.$$

Error estimates / bounds on the error in Trapezoidal & Simpson's Rules.

THEOREM

p. 498

TRAPEZOIDAL RULE

If
- f'' is continuous and
- $\underbrace{\forall x}_{\text{for all } x} \underbrace{\in}_{\text{in}} \underbrace{[a,b]}_{\text{the closed interval}}$, $|f''(x)| \leq \underbrace{M}_{\text{constant}}$

Then:
- Error in trapezoid approximation with $(n+1)$ many points / n many trapezoids

satisfies $|E_T| \leq \dfrac{M(b-a)^3}{12n^2}$,

where $E_T := \int_a^b f(x)\,dx - \left(\begin{array}{c}\text{Trapezoidal}\\ \text{approx.}\end{array}\right)$

- IF
  - f^{IV} is continuous
  - $\forall x \in [a,b]$, $|f^{IV}(x)| \leq M$

THEN $$|E_S| \leq \frac{M(b-a)^5}{180 n^4}, \quad \text{where}$$

$$E_S := \int_a^b f(x)\,dx - \begin{pmatrix} \text{Simpson approx.} \\ \text{with } n+1 \text{ points} \\ \text{or } n \text{ intervals} \end{pmatrix}$$

FINAL EXAM REVIEW.

09 June	*The Natural Logarithm as an Integral*	Section 7.1
14 June	*Exponential Growth and Decay*	Section 7.2
16 June	*Basic Integration Formulas, Integration by Parts*	Sections 8.1–8.2
21 June	*Trigonometric Integrals, Trigonometric Substitutions*	Sections 8.3–8.4
23 June	*Integration by Partial Fractions*	Section 8.5
28 June	*Numerical Integration*	Section 8.7
30 June	*Final Exam*	Sections 7.1–7.2, 8.1–8.5, 8.7

Like the midterm exam, the final ...

- Will be 2 hours long
- 90 total points (+10 bonus)
- No calculators
- One sheet of notes (8.5×11, front & back)

Also like the midterm, to study, you should review...

- Written homework
- WeBWork
- Suggested problems
- "Questions to guide your Review"
- Lecture notes
- Textbook.

7.1 : Log as integral

- Def: the natural log: $\ln(x) = \int_1^x \frac{1}{t}\,dt$
- Def: e : $\ln(e) = 1$
- The derivative $\frac{d}{dx}\left[\ln(x)\right]$
- The graph & range of $\ln(x)$
- The integral $\int \frac{1}{u}\,du$
- The inverse function of $\ln(x)$ & the number e
- $e^{\ln(x)} = x = \ln(e^x)$
- the deriv. & integral of e^x
- Laws of exponents $e^{a+b} = e^a e^b$

  $e^{ab} = (e^a)^b = (e^b)^a$

* Laws of logs $\ln(ab) = \ln(a) + \ln(b)$

$\ln(x^r) = r\ln(x)$, where $r \in \mathbb{Q}$.

- General exponential a^x } derivs & integrals — know!

- Logarithms base a ∫

Problems:
- Integrals
- IVPs

7.2: Exponential change (growth, decay) & Separable ODEs.

- Exponential change: $\begin{cases} \frac{dy}{dt} = ky, & k\text{ - const.} \\ y(0) = y_0 \end{cases}$

  $\exists$ solution to this ODE

- Separable diff'l eq'ns $\begin{cases} \frac{dy}{dx} = g(x)H(y) \\ \text{(init. cond.)} \end{cases}$

- Separation (divide by $H(y)$ (if $H(y) \neq 0$) and integrate both sides w.r.t. x)

- Unchecked population growth
- Radioactivity – the half-life
- Heat transfer – Newton's Law of Cooling

* Compounding interest (HW4)

Probs:
- Verifying sol'ns to ODEs
- Solving IVPs (by separat'n)

Ch. 7: Questions to guide Review : # 1–7

① Explain 7.1, re: $\ln(x)$

② $\ln(x)$ is the antideriv. of what? Give examples.

③ $\int \tan x \, dx = ?$ $\quad \int \sec x = ?$ $\quad \int \cot x \, dx = ?$ $\quad \int \csc x \, dx = ?$; why are we asking now?

④ Explain 7.1, re: e^x

⑤ General logarithms & exponentials

⑥ Solving separable 1^{st} order ODEs

⑦ Explain 7.2 re: exponential change.

8.1 : Using basic integrat'n formulas.

- TABLE 8.1 (!!!)

- u-substitution

- Completing the square

- Trig. formulas (given & used in 8.3 as well)

- $$\int_0^{\pi/4} \frac{dx}{1-\sin x} = \int_0^{\pi/4} \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx$$

  $= 1$

  "multiplying by 1"

- Long polynomial division

- $$\int \frac{3x+2}{\sqrt{1+x^2}}\,dx = \int \frac{3x}{\sqrt{1+x^2}}\,dx + \int \frac{2}{\sqrt{1+x^2}}\,dx$$

- odd/even fns. on symmetric intervals

Probs: Know how to use the techniques from the examples & the basic formulas in Tab. 8.1

8.2 : Integration by Parts

- Product rule in integral form
- FORMULA : $\int u \, dv = uv - \int v \, du$
- $\int \ln x \, dx$
- Repeated int. by parts, e.g. $\int e^x \cos x \, dx$
- Express $\int (\cos x)^n \, dx$ as an integral of a lower power of $\cos x$.
- Definite integrals by parts.

Probs:
- Integration by parts
- Using substitution

8.3 : Trig. integrals

- $\int (\sin x)^m (\cos x)^n \, dx$

$$\left.\begin{array}{l} \sin^2 x + \cos^2 x = 1 \\ \sin^2 x = \dfrac{1-\cos(2x)}{2} \end{array}\right\} \Rightarrow \cos^2 x = \frac{1+\cos(2x)}{2}$$

- $\int (\tan x)^m (\sec x)^n \, dx$

$$\tan^2 x = \sec^2 x - 1$$

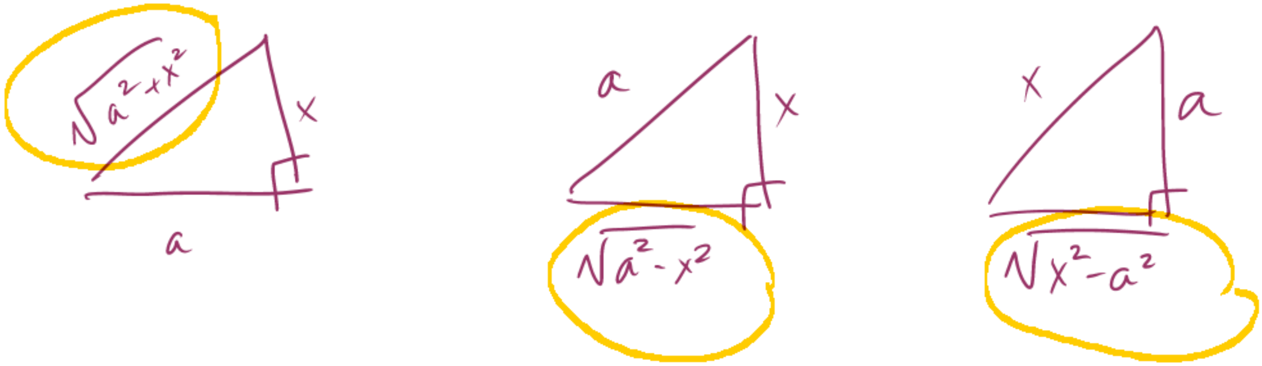
- Products of sines & cosines

p. 473
$$\left\{\begin{array}{l} \sin(mx)\sin(nx) = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] \\ \cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right] \\ \sin(mx)\cos(nx) = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right] \end{array}\right.$$

<u>Probs</u>: Know the techniques from examples

7.4: Trig. sub.

- 3 basic triangles



- Procedure: p. 476

Probs: • Using trig. sub. • IVPs

8.5: Partial Fractions

- Method of partial fracs. when $\frac{f(x)}{g(x)}$ is proper
  p. 481 (also, lecture notes)

- linear factors only
- repeated lin. factors
- quadratic factors
- repeated quad. factors
- long poly. division (improper frac'n to begin with)

- "Heaviside cover-up"

Probs:
- Expanding quotients
- (Non) repeated (irreducible quad. / linear) factors
- Improper fracs
- IVPs & integrals

8.7: Numerical Integration

- Trapezoidal
- Simpson's rule
- (error bounds/estimates

Probs:
- Estimating integrals
- —"— the # of subintervals required for a particular resolution/error
- Numerical/discrete data

Questions to guide review.

1–8, 11