Lecture 2: Thursday, 19 May.

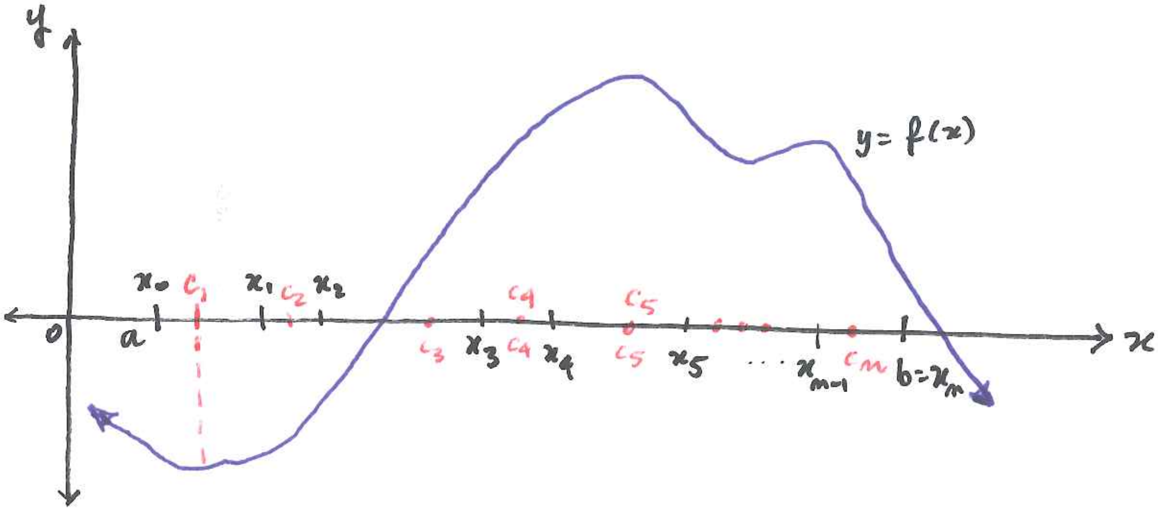
Announcements / Assignments.

- WW1 (from Tuesday's class) due Friday 11:59 p.m.
- WW2 (tonight's class) due Monday 11:59 p.m.
- HW1 due Monday 11:59 p.m.
- Syllabus / Background quiz on myWPI – complete ASAP!
- In-class students: please find NEXT WEEK's lectures (i.e., Tuesday 24 May & Thursday 26 May) ONLINE ONLY – I will be travelling and won't be on campus! (I will post these lectures before leaving).

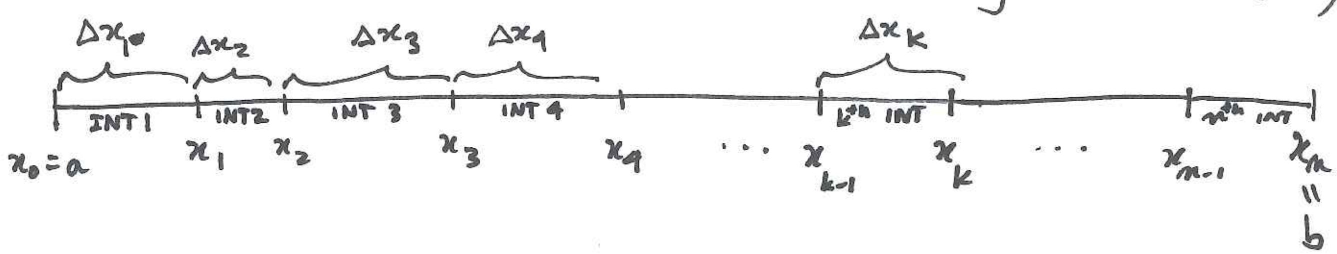
... Questions?

5.2, ct'd: RIEMANN SUMS.

Begin with an arbitrary bounded function f on the closed interval $[a, b]$. (f may have negative values as well as positive ones.)



• Subdivide $[a, b]$ into intervals (not necessarily equal length)



• Call interval lengths $\Delta x_k := x_k - x_{k-1}$.

• Within each (closed) interval, choose a point $c_k \in [x_{k-1}, x_k]$.

"Upper" sums: $c_k = x_{k-1}$ "LEFT ENDPOINTS"

"Lower" sums: $c_k = x_k$ "RIGHT ENDPOINTS"

"Midpoint" sums: $c_k = \frac{x_k + x_{k-1}}{2}$ (mid pt.)

This is where to evaluate the function that gives each "rectangle" height!

- Draw rectangles (optional)
- Area of k^{th} ~~each~~ rectangle is : $\Delta x_k \cdot f(c_k)$
- Sum these areas:

$$S_P := \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

is called ~~the~~ a Riemann sum for f on the interval $[a,b]$.

Can have different Riemann sums depending on our partition "P" of $[a,b]$ into subintervals, and on our choice of the c_k.

Recall: c_k = left-hand endpoint of interval
c_k = right – –
c_k = midpoint of interval
} all valid choices!

Generally, accuracy increases as $n \to \infty$ (for constant-length intervals).

If intervals in a partition are not of constant length, then we introduce the notion of the NORM:

> **DEF.** For a given partition P of a given interval I, the **NORM** of P is the largest of all subinterval widths, and is written $\|P\|$.

Example. Consider the function $f(x) := x^2$ on the interval $[0,2]$.

6, p.314

- A partition of $[0,2]$ is $P := \{0, 0.2, 0.6, 1, 1.5, 2\}$

The norm of this partition is the width of the largest subinterval.

$$\Delta x_1 = 0.2 - 0 = 0.2$$

$$\Delta x_2 =$$

$$\Delta x_3 =$$

$$\Delta x_4 =$$

$$\Delta x_5 =$$

So the norm of this partition is ________.

Example, ct'd.

So we have our partition $P := \{0, 0.2, 0.6, 1, 1.5, 2\}$ with norm ______ .

- Let c_k, for $k = 1, 2, 3, 4, 5$, be the left-hand endpoint of each interval. So $c_1 = 0$, $c_2 = 0.2$, ... $c_5 = 1.5$.

Then

$$A \approx \sum_{k=1}^{5} \Delta x_k f(c_k) \quad , \text{where } f(x) = x^2 .$$

$$= \Delta x_1 c_1^2 + \Delta x_2 c_2^2 + \Delta x_3 c_3^2 + \Delta x_4 c_4^2 + \Delta x_5 c_5^2$$

$$= 0.2(0)^2 + 0.4(0.2)^2 + 0.4(0.6)^2 + 0.5(1)^2 + 0.5(1.5)^2$$

$$= \frac{2}{5}\left(\frac{1}{5}\right)^2 + \frac{2}{5}\left(\frac{3}{5}\right)^2 + \frac{1}{2} + \frac{1}{2}\left(\frac{3}{2}\right)^2$$

$$= \frac{2}{125} + \frac{18}{125} + \frac{1}{2} + \frac{9}{8}$$

$$= \frac{20}{125} + \frac{13}{8} \approx \text{______} .$$

DEF. Suppose $f(x)$ is defined on the closed interval $[a,b]$. We say J is <u>the definite integral of f over $[a,b]$</u>, and that J is <u>the limit of the Riemann sums</u> $\sum_{k=1}^{n} \Delta x_k f(c_k)$ if:

For all $\varepsilon > 0$, there exists $\delta > 0$ such that:

For every partition P of $[a,b]$ with $\|P\| < \delta$ and any choice of $c_k \in [x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon .$$

This should remind us of the ε-δ definitions we saw in Calc I (limits / continuity).

All the definition says is that <u>if</u> the Riemann sums (for any c_k) have a limit <u>as the norm of the partition approaches 0</u>, then that limit is defined as the definite integral of the function:

$$J = \lim_{\|P\| \to 0} \left(\sum_{k=1}^{n} f(c_k)\Delta x_k \right).$$

The symbol we use for the definite integral comes (roughly) from the sigma notation for Riemann sums.

$$\lim_{\|P\|\to 0}\left(\sum_{k=1}^{n} f(c_k)\,\Delta x_k\right) =: \int_a^b f(x)\,dx$$

upper limit of integration

function to integrate

differential

INTEGRAL SIGN (a big "S" for "sum")

lower limit of integration

The definite integral is read out loud as:

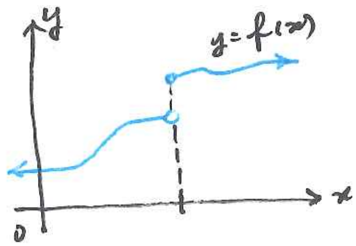
" the integral from a to b of f with respect to x".

> **DEF.** If f has a definite integral over $[a,b]$, then we say f is <u>integrable</u>.

- Not every function is integrable!
  (that means there are some functions whose Riemann sums do not converge)

THM. 1, p.318 If a function f is continuous over $[a,b]$, or if it has at most finitely many "jump" discontinuities there, then f is integrable over $[a,b]$.

Recall: a "jump" discontinuity looks like:



Examples.
- $f(x) = 1$ is/is not integrable over $[0,1]$.
- $f(x) = \begin{cases} 1, & \text{if } x \geq 1/2 \\ 0, & \text{if } x < 1/2 \end{cases}$ is/is not integrable over $[0,1]$.
- $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$ is/is not integrable over $[0,1]$.

5.3, ct'd.

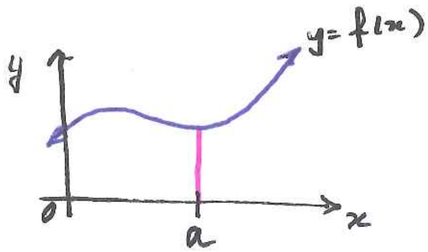
Definite integrals follow the rules in Table 5.6:

1. Order of Integration

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

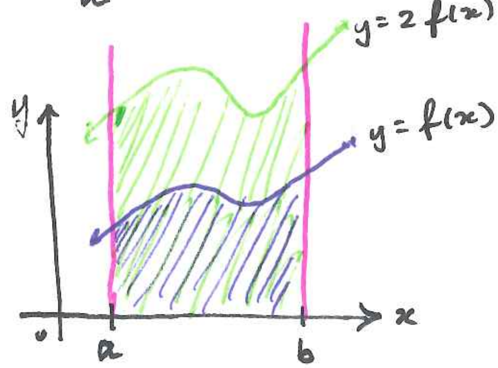
2. Zero-width interval.

$$\int_a^a f(x)\,dx = 0$$



3. Constant multiple.

$$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$



4. Sum and difference.

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

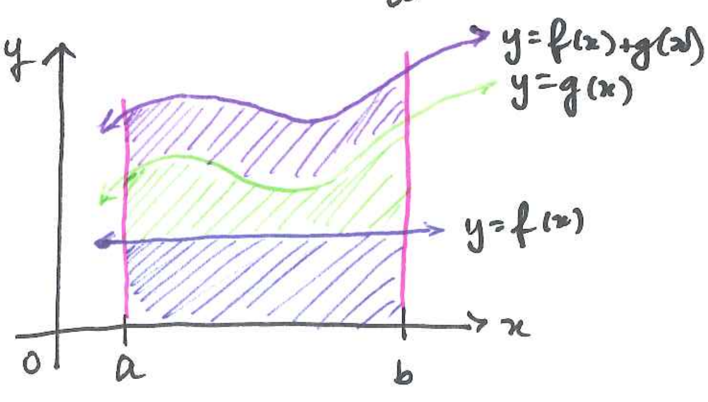
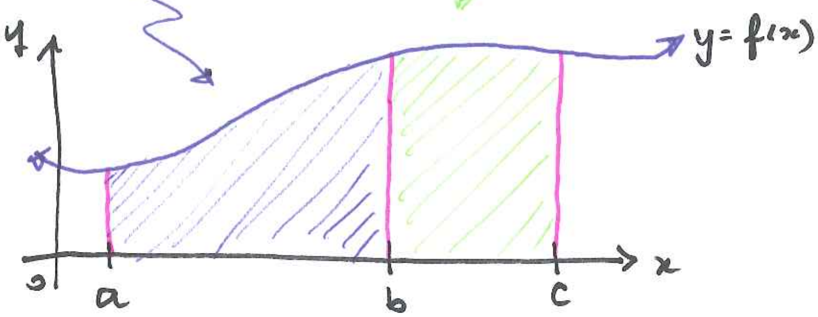


Table 5.6: Integration rules, ct'd.

5. Additivity.

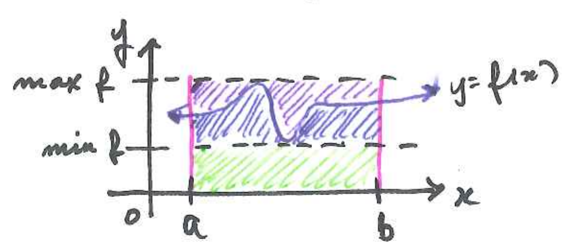
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



6. Max-minimum inequality.

If f has maximum value ($\max f$) and minimum value ($\min x$) over $[a,b]$, then:

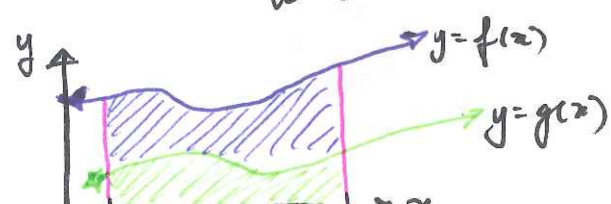
$$(\min f)\cdot(b-a) \leq \int_a^b f(x)\,dx \leq (\max f)\cdot(b-a)$$



7. Domination.

If $f(x) \geq g(x)$ for all $x \in [a,b]$, then:

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$



The definite integral rules #2-7 can be proved (see p. 321).

Examples. 2, p.321 Suppose $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$, $\int_{-1}^{1} g(x)\,dx = 7$.

Then...

Rule 1 (order of integration)

$$\int_{1}^{-1} f(x)\,dx = -\int_{-1}^{1} f(x)\,dx = -5.$$

$$\int_{4}^{1} f(x)\,dx = \underline{\qquad\qquad\qquad}$$

Rule 3 and Rule 4 (const. mult., sum/diff.)

$$\int_{-1}^{1} 3f(x) - 2g(x)\,dx = 3\int_{-1}^{1} f(x)\,dx - 2\int_{-1}^{1} g(x)\,dx$$

$$= 3(5) - 2(7)$$

$$= 15 - 14$$

$$= 1.$$

Rule 5 (additivity).

$$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$$

$$= 5 + (-2)$$

$$= 3.$$

Applications of the definite integral.

- Computing the area under a curve.

[IMPORTANT: If $y=f(x)$ is nonnegative and integrable over $[a,b]$,
Then the area under the curve $y=f(x)$ over $[a,b]$ IS the integral of f from a to b.]

!!

Example: 4, p. 322 Compute $\int_0^b x\,dx$ to find the area A under $y=x$ over $[0,b]$.
(Suppose $b>0$.)

Since $f(x) := x$ is continuous over $[0,b]$, THEOREM 1 says it is integrable there.

Then we subdivide $[0,b]$ into n many subintervals, each of width $\frac{b}{n}$. (What is the norm of the partition? ____)

Choose the c_k to be right-hand endpts. (could be left, etc.)

So $P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n} = b\right\}$ and $c_k = \frac{kb}{n}$, $k = 1, 2, \dots, n$.

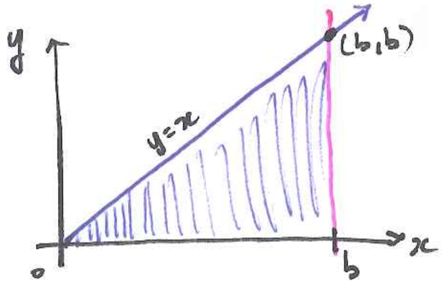
and $$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k = \frac{b^2}{n^2}\,\frac{n(n+1)}{2} =$$

$$= \frac{b^2}{2}\left(1+\frac{1}{n}\right).$$

And $\lim_{n\to\infty} \frac{b^2}{2}\left(1+\frac{1}{n}\right) =$ ______ . So $\int_0^b x\,dx =$ ______ .

Another way of doing Example 4:

The curve $y = x$ over $[0,b]$ looks like:



... it is a right triangle! Moreover, it is a right ________ triangle (base = height).

So area is: ______________________.

Example 4 gave us a formula; for $f(x) = x$,

$$\int_0^b f(x)\,dx = \frac{b^2}{2}$$

What is $\displaystyle\int_a^b f(x)\,dx$?

$$\int_a^b f(x)\,dx = \int_0^b f(x)\,dx - \int_0^a f(x)\,dx.$$ when $f(x) = x$:

$$=$$

5.4: The Fundamental Theorem of Calculus.

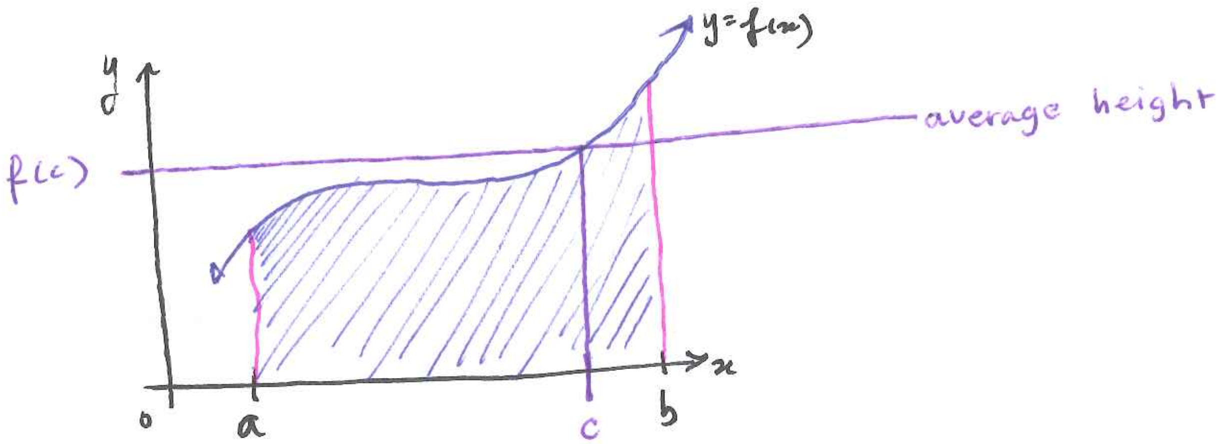
A theorem that is used in the proof of the Fundamental Theorem of Calculus is the MEAN VALUE THEOREM:

THEOREM (MVT, p. 328) If f is continuous on $[a,b]$, Then at some point $c \in [a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

"A continuous function on a closed interval attains its mean value."

(-Sound familiar? -)



This is a very important theorem!

Even though we will not prove the Fundamental Theorem of Calculus in this course, the Mean Value Theorem is important in its own right — and you should be able to state it (on an exam?).

THEOREM. FUNDAMENTAL THM OF CALCULUS (PART I)

If: f is continuous on $[a,b]$, ~~then~~

Then: the function $F(x) := \int_a^x f(t)\,dt$ is also continuous on $[a,b]$, and it is an antiderivative of $f(x)$. That is:

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Examples. Use the FTC to find $\frac{dy}{dx}$ if:
2, p. 330

(a) $y = \int_a^x t^3 + 1\,dt$ (b) $y = \int_x^5 3t \sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$ (d) $y = \int_{1+3x^2}^4 \frac{1}{2+e^t}\,dt$.

Sol. (a) $\frac{dy}{dx} = \frac{d}{dx}\left[\int_a^x t^3 + 1\,dt\right] =$ ________.

(Here, $F(x) = \int_a^x f(t)\,dt$, so $f(x) =$ ________.)

Examples (2), ct'd.

(b) $$\frac{dy}{dx} = \frac{d}{dx}\left[\int_x^5 3t \sin t \, dt\right]$$

$$= \frac{d}{dx}\left[-\int_5^x 3t \sin t \, dt\right]$$

$$= -\frac{d}{dx}\left[\int_5^x 3t \sin t \, dt\right]$$

$$= -3x \sin x.$$

(c) $$\frac{dy}{dx} = \frac{d}{dx}\left[\int_1^{x^2} \cos t \, dt\right].$$

Let $g(u) := \int_1^u \cos t \, dt$ and $u(x) := x^2$.

Then $y = g(u(x))$, and to differentiate, we use

the ________ rule:

$$\frac{dy}{dx} =$$

Examples (2), ct'd.

(d) $$y = \int_{1+3x^2}^{4} \frac{1}{2+e^t}\, dt$$

THEOREM.
FUNDAMENTAL THM. OF CALCULUS PART II

If: f is continuous over $[a,b]$ and
F is any antiderivative of f on $[a,b]$,

Then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The two parts of the Fundamental Theorem of Calculus give us a way of connecting our study of antiderivatives to our study of integrals!

... and they also give us a "natural" way of defining the "INDEFINITE INTEGRAL":

DEF. The collection of all antiderivatives of f is called the INDEFINITE INTEGRAL of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

Part II of the FTC also gives us a *much* easier way of computing *definite* integrals: To find $\int_a^b f(x)\,dx$:

1. Find an antiderivative F of f
2. Compute $F(b) - F(a)$.

... much better than taking limits of Riemann sums, right ??

Example: Compute:
3, p. 332

(a) $\displaystyle\int_0^{\pi} \cos x \; dx \quad =$

(b) $\displaystyle\int_1^4 \frac{3}{2}\sqrt{x} - \frac{4}{x^2} \quad dx \quad =$

Applications of integration / FTC.

THEOREM "Net Change Theorem"
5, p.333

The net change in a differentiable function $F(x)$ over an interval $[a,b]$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\, dx.$$

- If $c(x)$ is the cost of producing x units of something, Then $c'(x)$ is the marginal cost, and

$$\int_{x_1}^{x_2} c'(x)\, dx = c(x_2) - c(x_1).$$

This is the cost of increasing production from x_1 to x_2 units.

- If an object with position fn. $s(t)$ moves along a line, its velocity is $v(t) := s'(t)$.

Then

$$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1).$$

So the integral of the velocity is the displacement.