Lecture 3 : Substitution Method (Indefinite Integrals)

Announcements / Assignments.

- Homework 1 due Monday, May 23 at 11:59 p.m. — submit to myWPI
- Homework 2 due Monday, May 30 at 11:59 p.m. — submit to myWPI
- Webwork 2 due Monday, May 23 at 11:59 p.m.
- Webwork 3 due Friday, May 27 at 11:59 p.m.

- No in-person office hours on Tuesday, May 24 or on Thursday, May 26 !!

  Please e-mail emkiley@wpi.edu with any questions or to make a meeting online with me.

Today's Lecture:

Section 5.5: The Substitution Method for Indefinite Integrals.

Section 5.5: Substitution for Indefinite Integrals.

Recall: The indefinite integral of a function f with respect to x is the set of all antiderivatives of f, symbolized by $\int f(x)\,dx$.

Since any two antiderivatives of f differ by only a constant (this was Theorem 8 from 4.8), we can write

$$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any antiderivative of f, and where C is an arbitrary constant.

IMPORTANT: An indefinite integral is a FUNCTION.
A definite integral is a NUMBER.

Antidifferentiation technique: SUBSTITUTION.

Recall: the Chain rule for derivatives (Section 3.6)

If $y = y(u)$ and $u = g(x)$,

Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, where $\dfrac{dy}{du}$ is evaluated at $u = g(x)$.

e.g., $\dfrac{d}{dx}\left[\dfrac{u^{n+1}}{n+1}\right] = u^n \dfrac{du}{dx}$.

This means that an antiderivative of $u^n \dfrac{du}{dx}$ is __________.

And therefore, $\displaystyle\int u^n \frac{du}{dx}\, dx =$ _______________.

But also, $\displaystyle\int u^n\, du =$ _______________.

This may lead us to wonder whether it is <u>always</u> possible to substitute "$\dfrac{du}{dx}\, dx \overset{?}{=} du$".

Leibniz wondered the same thing!

... this insight leads to the Substitution Method,

Example. Find the (indefinite) integral
1, p. 340

$$\int (x^3+x)^5 (3x^2+1)\, dx .$$

Let $u(x) := x^3 + x$. Then $du := \frac{du}{dx}\, dx$

$$= \frac{d}{dx}\left[x^3+x\right] dx$$

$$= (3x^2+1)\, dx .$$

So the integral becomes:

$$\int \underbrace{(x^3+x)}_{u(x)}{}^5\, \underbrace{(3x^2+1)\, dx}_{du} = \int u^5\, du .$$

Now, $\int u^5\, du =$ ________________

and we substitute $u(x) = x^3 + x$ back in, so finally,

$\int (x^3+x)^5 (3x^2+1)\, dx =$ ____________________ .

Example 2, p. 340

Find $\int \sqrt{2x+1}\, dx$.

Let $u(x) := 2x+1$. Then $du = \frac{du}{dx}\, dx$

$= \frac{d}{dx}\left[2x+1\right] dx$

$= 2\, dx$.

Then $dx = \_\_\_ du$, so the integral becomes:

$$\int \underbrace{\sqrt{2x+1}}_{u=2x+1}\, \underbrace{dx} = \int \_\_ \sqrt{u}\, du = \_\_ \int \sqrt{u}\, du$$

$$= \_\_\_ \int u^{1/2}\, du$$

$$= \_\_\_ \cdot \frac{2}{3} u^{3/2} + C$$

$$= \_\_\_ \cdot \frac{2}{3} (2x+1)^{3/2} + C.$$

Check: $\frac{d}{dx}\left[(2x+1)^{3/2} + C \right] = \frac{3}{2}(2x+1)^{1/2} \cdot \frac{d}{dx}(2x+1)$

$= \frac{3}{2} \cdot 2 \cdot (2x+1)^{1/2}$

$= \sqrt{2x+1}$. ✓

Example 3 Find $\int \sec^2(5x+1)\cdot 5\,dx$.

p. 341

Let $u := 5x+1$. Then $du = \frac{du}{dx}\,dx = \frac{d}{dx}[5x+1]\,dx$

$= 5\,dx$,

so the integral becomes:

$$\int \sec^2(5x+1)\cdot 5\,dx = \int \sec^2 u\,du. \quad \text{(integrate/antidifferentiate)}$$

$$= \tan u + C \quad \text{(back substitute)}$$

$$= \tan(5x+1) + C.$$

Check: $\frac{d}{dx}\left[\tan(5x+1)+C\right] = \sec^2(5x+1)\,\frac{d}{dx}(5x+1)$

$= \sec^2(5x+1)\cdot 5$ ✓.

Example 4, p. 341 Find $\int \cos(7\theta+3)\, d\theta$.

Let $u := 7\theta + 3$, so $du = \frac{du}{d\theta}\, d\theta = \frac{d}{d\theta}(7\theta+3)\, d\theta = 7\, d\theta$.

Then $d\theta = \frac{1}{7}\, du$, and the integral becomes:

$$\int \cos(7\theta+3)\, d\theta = \int \cos u \cdot \frac{1}{7}\, d\cancel{\theta}u = \frac{1}{7}\int \cos u \, du =$$

$$= \frac{1}{7} \sin u + C$$

$$= \frac{1}{7} \sin(7\theta+3) + C.$$

Check: $\frac{d}{d\theta}\left[\frac{1}{7}\sin(7\theta+3)\right] = \frac{1}{7}\cos(7\theta+3)\frac{d}{d\theta}\left[7\theta+3\right]$

$$= \frac{1}{7}\cos(7\theta+3)\cdot 7$$

$$= \cos(7\theta+3). \quad \checkmark$$

Example. Find $\int x^2 e^{x^3}\,dx$.
5, p. 342

Let $u := x^3$, so $du = \frac{du}{dx}\,dx = \frac{d}{dx}[x^3]\,dx = 3x^2\,dx$.

Then $dx = \frac{1}{3x^2}\,du$, and the integral becomes:

$$\int x^2 e^{x^3}\,dx = \int e^u \left(\frac{x^2}{3x^2}\,du\right) = \int \frac{1}{3} e^u\,du$$

* NOTICE: integral contains ONLY u-terms!!

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3} e^u + C$$

$$= \frac{1}{3} e^{x^3} + C.$$

Check: $\frac{d}{dx}\left[\frac{1}{3} e^{x^3} + C\right] = \frac{1}{3} e^{x^3} \cdot \frac{d}{dx}[x^3]$

$$= \frac{1}{3} e^{x^3}(3x^2)$$

$$= x^2 e^{x^3}. \checkmark$$

Example 6, p. 343 Find $\int x\sqrt{2x+1}\,dx$.

Let $u := 2x+1$, as before. Then $du = \frac{du}{dx}\,dx = 2\,dx$,

and so $dx = \frac{1}{2}\,du$. Substitute:

$$\int x\sqrt{2x+1}\,dx = \int x\sqrt{u}\left(\tfrac{1}{2}\,du\right)$$
$$= \frac{1}{2}\int x\sqrt{u}\,du.$$

PROBLEM !!!! An integral with both x and u makes little sense to us.

But observe...

If $u = 2x+1$, then $2x = u-1$, so $x = \frac{1}{2}(u-1)$.

That means...

$$\int x\sqrt{2x+1}\,dx = \frac{1}{2}\int x\sqrt{u}\,du = \frac{1}{2}\int\left(\tfrac{1}{2}(u-1)\right)\sqrt{u}\,du$$
$$= \frac{1}{4}\int (u-1)\sqrt{u}\,du$$
$$= \frac{1}{4}\int u\sqrt{u} - \sqrt{u}\,du$$
$$= \frac{1}{4}\int u^{3/2}\,du - \frac{1}{4}\int u^{1/2}\,du$$
$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2}\right) - \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C$$

(continued)

Example 6

ct'd.

So $\int x\sqrt{2x+1}\,dx = \frac{1}{4}\left(\frac{2}{5}u^{5/2}\right) - \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C$

$$= \frac{2}{20}u^{5/2} - \frac{2}{12}u^{3/2} + C$$

$$= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C$$

$$\cancel{= \frac{1}{2}u\left(\frac{1}{5}u^{3/2} - \frac{1}{3}u^{1/2}\right) + C}$$

$$\cancel{= \frac{1}{2}(2x+1}$$

(not helpful)

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.$$

Check: $\frac{d}{dx}\left[\frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C\right] =$ *

$$= \frac{1}{10}\left[\frac{5}{2}(2x+1)^{3/2}(2)\right] - \frac{1}{6}\left[\frac{3}{2}(2x+1)^{1/2}(2)\right]$$

$$= \frac{1}{2}(2x+1)^{3/2} - \frac{1}{2}(2x+1)^{1/2}$$

$$= \frac{1}{2}\sqrt{2x+1}\left[(2x+1)-1\right]$$

$$= \frac{1}{2}\sqrt{2x+1}\,(2x)$$

$$= x\sqrt{2x+1} \quad \checkmark$$

Example 7. p. 343 (c) Find $\int \tan x \, dx$.

Well, we know $\tan(x) = \dfrac{\sin x}{\cos x}$, so

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Let $u := \cos x$, so $du = \dfrac{du}{dx} dx = \dfrac{d}{dx}[\cos x] \, dx$
$= -\sin x \, dx$.

The integral becomes:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int -\frac{1}{u} \, du$$

$$= -\ln|u| + C$$

ABSOLUTE VALUE SIGNS CRUCIAL !!

(you will lose points for forgetting them)

$$= -\ln|\cos x| + C.$$

Recall: $-\ln(a) = \ln\left(\frac{1}{a}\right)$ for all a.

So $\int \tan x \, dx = -\ln|\cos x| + C$

$$= \ln\left(\frac{1}{|\cos x|}\right) + C$$

$$= \ln\left|\frac{1}{\cos x}\right| + C$$

$$= \ln|\sec x| + C.$$

Example 8, p. 344 (a) $\int \frac{dx}{e^x + e^{-x}}$. Multiply by $\frac{e^x}{e^x} = 1$.

$$= \int \frac{e^x}{e^{2x}+1}\, dx$$

$$= \int \frac{e^x}{(e^x)^2+1}\, dx \qquad \text{because } a^{b\cdot c} = (a^b)^c = (a^c)^b \text{ for all } a, b, c$$

Let $u := e^x$, so $du = \frac{du}{dx}\, dx = \frac{d}{dx}(e^x)\, dx = e^x\, dx$.

Then $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\, dx}{(e^x)^2+1} = \int \frac{du}{u^2+1}$.

Recall: derivatives of the inverse trig. functions (on endsheets of textbook)

$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$, similar formulas for other inverse trig. fns.

So $\int \frac{dx}{e^x+e^{-x}} = \int \frac{du}{u^2+1} = \arctan(u) + C$

$$= \arctan(e^x) + C.$$

Example 9, p. 344 Find $I := \int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int 2z(z^2+1)^{-1/3}\,dz.$

Let $u := z^2+1$, so $du = \frac{du}{dz}\,dz = \frac{d}{dz}(z^2+1)\,dz = 2z\,dz.$

Then $I = \int 2z(z^2+1)^{-1/3}\,dz = \int u^{-1/3}\,du$

$$= \left(\frac{3}{2}\right)u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C.$$

Alternatively (and more difficult / unnecessary):

Let $u := (z^2+1)^{1/3}$, so $u^3 = z^2+1$ and $3u^2\,du = 2z\,dz$.

Then $I = \int \frac{3u^2\,du}{u} = \int 3u\,du$

$$= 3\int u\,du$$

$$= \frac{3}{2}u^2 + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C.$$