Lecture 4.

Announcements/Assignments.

- Homework 1 due Monday, May 23 at 11:59 p.m.
- Webwork 2 ———— " ————
- Webwork 3 due Friday, May 27 at 11:59 p.m.
- Homework 2 due Monday, May 30 at 11:59 p.m.
- Webwork 4 ———— " ————

- No office hours in person on Tuesday & Thursday May 24 & 26 – email Erin for help:

  emkiley@wpi.edu

TODAY: Section 5.6, Substitution for Definite Integrals.

Section 5.6: Substitution for Definite integrals, the Area between curves.

Substitution for definite integrals requires that we also transform the bounds (limits) of integration.

Example. 1, p.347 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\, dx$.

Let $u(x) := x^3+1$. Then $du = \frac{du}{dx}\, dx = \frac{d}{dx}[x^3+1]\, dx = 3x^2\, dx$,

and $u(-1) = (-1)^3+1 = -1+1 = 0$

$u(1) = (1)^3+1 = 1+1 = 2$.

So the integral becomes:

$$\int_{x=-1}^{1} 3x^2\sqrt{x^3+1}\, dx = \int_{u=0}^{2} \sqrt{u}\, du$$

$$= \int_0^2 u^{1/2}\, du$$

$$= \frac{2}{3} u^{3/2} \Big|_{u=0}^{2}$$

$$= \frac{2}{3}(2)^{3/2} - \frac{2}{3}(0)^{3/2}$$

$$= \frac{2}{3}(2)^{3/2}$$

$$= \frac{2}{3} \cdot 2\sqrt{2} = \frac{4}{3}\sqrt{2}.$$

Example

1, p. 347, ct'd.

Another way is to use the indefinite integral...

Observe: $\int 3x^2\sqrt{x^3+1}\,dx$ (indefinite!) can be evaluated by substitution:

Let $u := x^3+1$, so $du = \frac{du}{dx}\,dx = \frac{d}{dx}\left[x^3+1\right]dx = 3x^2\,dx$.

Then $\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$.

$= \frac{2}{3}(x^3+1)^{3/2} + C$

So
$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big|_{x=-1}^{1}$$

$$= \frac{2}{3}\left((1)^3+1\right)^{3/2} - \frac{2}{3}\left((-1)^3+1\right)^{3/2}$$

$$= \frac{2}{3}(1+1)^{3/2} - \frac{2}{3}(-1+1)^{3/2}$$

$$= \frac{2}{3}(2)^{3/2}$$

$$= \frac{2}{3}\cdot 2\sqrt{2} = \boxed{\frac{4}{3}\sqrt{2}.}$$

They match!

Generally, you can use either method.

Example 2, p. 348

(a) $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta$.

Easiest way is to simplify first:

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta \, d\theta = \int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin\theta}\left(\frac{1}{\sin^2\theta}\right) d\theta = \int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin^3\theta}\, d\theta.$$

Method 1 (transforming the bounds)

Let $u(\theta) := \sin\theta$, so $du = \frac{du}{d\theta}\, d\theta = \frac{d}{d\theta}[\sin\theta]\, d\theta = \cos\theta \, d\theta$,

and $u(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$,

$u(\pi/2) = \sin(\pi/2) = 1$.

Then the integral becomes:

$$\int_{\theta=\pi/4}^{\pi/2} \frac{\cos\theta}{\sin^3\theta}\, d\theta = \int_{u=\frac{\sqrt{2}}{2}}^{1} \frac{du}{u^3} = \int_{\sqrt{2}/2}^{1} u^{-3}\, du =$$

$$= -\frac{1}{2}u^{-2}\Big|_{u=\sqrt{2}/2}^{1} = \frac{-1}{2u^2}\Big|_{\sqrt{2}/2}^{1} =$$

$$= -\frac{1}{2\cdot 1^2} - \left(\frac{-1}{2\cdot(\sqrt{2}/2)^2}\right) = -\frac{1}{2} + \frac{1}{2(1/2)}$$

$$= -\frac{1}{2} + 1 = \boxed{\frac{1}{2}}.$$

Example 2(a) p.348, ct'd

Alternatively, use the indefinite integral:

$$\int \cot\theta \csc^2\theta \, d\theta = \int \frac{\cos\theta}{\sin^3\theta} \, d\theta$$, and use the same

substitution: $u(\theta) = \sin\theta$ and $du = \frac{du}{d\theta} \, d\theta = \cos\theta \, d\theta$,

so the indef. integral becomes:

$$\int \frac{\cos\theta}{\sin^3\theta} \, d\theta = \int \frac{du}{u^3} = \int u^{-3} \, du = -\frac{1}{2} u^{-2} + C = \frac{-1}{2u^2} + C.$$

$$= \frac{-1}{2\sin^2\theta} + C$$

Therefore, the def. integral:

$$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \left. \frac{-1}{2\sin^2\theta} \right|_{\theta=\pi/4}^{\pi/2}$$

$$= -\frac{1}{2\sin^2(\pi/2)} - \left(\frac{-1}{2\sin^2(\pi/4)} \right)$$

$$= \frac{-1}{2(1)^2} - \left(\frac{-1}{2(\sqrt{2}/2)^2} \right)$$

$$= -\frac{1}{2} + \frac{1}{2(1/2)}$$

$$= -\frac{1}{2} + 1 = \boxed{\frac{1}{2}.}$$ The same!

Example 2, p. 348 (b) $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x \, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx.$

Method 1.

Let $u(x) := \cos x$, so $du = \frac{du}{dx}\,dx = \frac{d}{dx}[\cos x]\,dx = -\sin x \, dx$

COSINE IS AN "EVEN" FN.

and $u(-\pi/4) = \boxed{\cos(-\pi/4) = \cos(\pi/4)} = \sqrt{2}/2$

$u(\pi/4) = \cos(\pi/4) = \sqrt{2}/2.$

So the integral becomes

$$\int_{x=-\pi/4}^{\pi/4} \tan x \, dx = \int_{x=-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx = \int_{u=\frac{\sqrt{2}}{2}}^{\sqrt{2}/2} \frac{-du}{u}.$$

Recall: "zero-width integral" $\displaystyle\int_a^a f(x)\,dx = 0.$

So $\displaystyle\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}/2} -\frac{du}{u} = 0$, and $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x \, dx = 0.$

Example 2(b) p. 348, ct'd. ...or could use indefinite integral:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \quad ; \quad \text{use same substitution:}$$

$$u(x) := \cos x \, , \quad \text{so} \quad du = \frac{du}{dx} dx = \frac{d}{dx}\left[\cos x\right] dx = -\sin x \, dx .$$

Then

$$\int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C,$$

so the definite integral is:

$$\int_{-\pi/4}^{\pi/4} \tan x \, dx = -\ln|\cos x| \Big|_{-\pi/4}^{\pi/4}$$

$$= -\ln|\cos \pi/4| - (-\ln|\cos(-\pi/4)|)$$

$$= -\ln|\sqrt{2}/2| + \ln|\sqrt{2}/2|$$

$$= \ln(\sqrt{2}/2)(-1+1)$$

$$= \ln(\sqrt{2}/2)(0)$$

$$= 0. \quad \checkmark$$

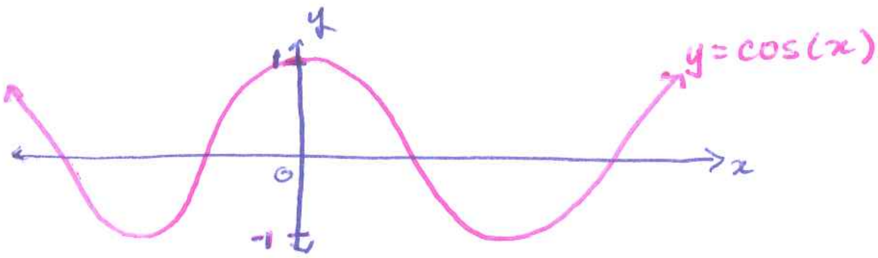
The last example was interesting: $\int_{-\pi/4}^{\pi/4} \tan x = 0$.

<u>Recall</u>: Even / odd functions.

- If for all x, $f(x) = f(-x)$,

  Then f is called an <u>EVEN</u> function,

  and the graph $y = f(x)$ is symmetric about the line $x = 0$; <u>Example</u>:

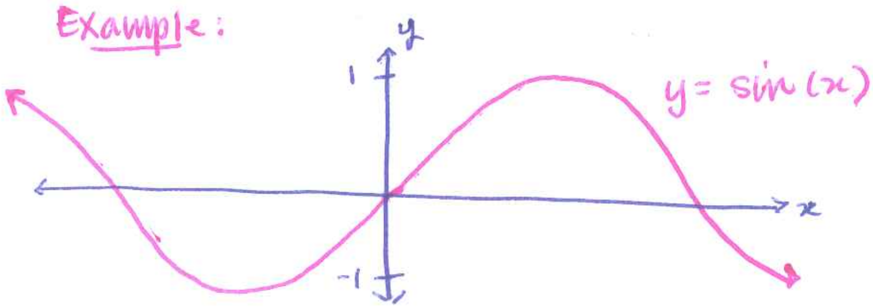


and $\cos(x) = \cos(-x)$ for all x

- If, for all x, $f(x) = -f(-x)$,

  Then f is called an <u>ODD</u> function;

<u>Example</u>:



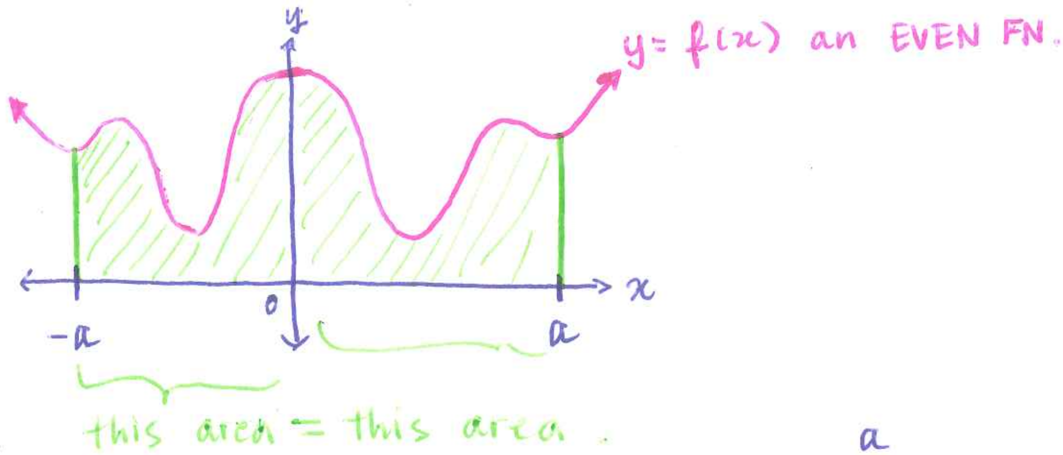
and $\sin(+x) = -\sin(-x)$ for all x.

[If you forget whether sin or cos is odd or even, check the unit circle or the map $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$.]

over a symmetric interval $[-a, a]$, the even and odd functions have special definite integrals.

Example

EVEN FN.



Intuition therefore says $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

for even functions. BUT DO NOT EVER TRUST A PICTURE.

... instead, compute:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$

Let $u = -x$, so $du = \frac{du}{dx}\,dx = \frac{d}{dx}(-x)\,dx = -dx$,

$u(0) = -0 = 0$ and $u(-a) = -(-a) = a$.

Then $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{u=0}^{a} f(-u)\,du + \int_{x=0}^{a} f(x)\,dx \longrightarrow$

So

$$\int_{-a}^{a} f(x)\,dx = \int_{u=0}^{a} f(-u)\,du + \int_{x=0}^{a} f(x)\,dx$$

and use $f(-u) = f(u)$, so

$$\int_{-a}^{a} f(x)\,dx = \int_{u=0}^{a} f(u)\,du + \int_{x=0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx.$$

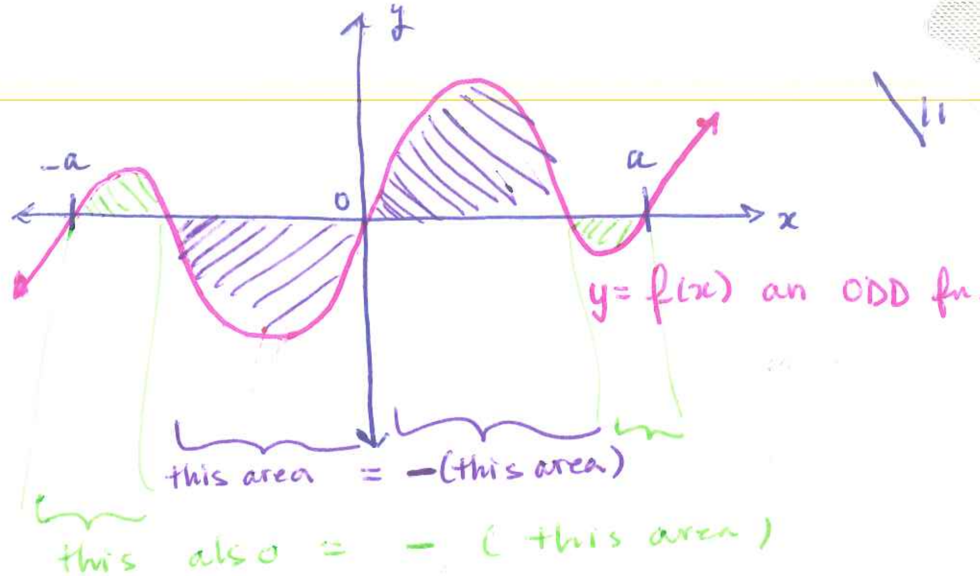
This proves that for an even function over a symmetric interval,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

... what about an <u>odd</u> function?

EXAMPLE

ODD FUNCTION



And intuition (plus our $\int_{-\pi/4}^{\pi/4} \tan x \, dx$ example) says $\int_{-a}^{a} f(x)\,dx = 0$.

BUT STILL NEVER TRUST A PICTURE!!

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

Let $u(x) := -x$, so $du = -dx$, $u(0) = 0$ and $u(-a) = a$.

Then $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$.

use the fact that $f(-u) = -f(u)$ (f was odd).

Then $\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx$

$$= 0.$$

In Summary:

THEOREM 8, p.349 Let f be continuous on $\underbrace{[-a, a]}_{\text{"symmetric interval"}}$.

Then...

- If f is <u>even</u>, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is <u>odd</u>, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 2(b), p. 348 REVISITED

$$\int_{-\pi/4}^{\pi/4} \tan(x)\,dx .$$

Notice that $\tan(x)$ is an odd function of x, since

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x).$$

Since $\int_{-\pi/4}^{\pi/4} \tan(x)\,dx$ is the integral of an odd function over a symmetric interval $[-\pi/4, \pi/4]$, the integral is zero:

$$\int_{-\pi/4}^{\pi/4} \tan x \,dx = 0 .$$

Example. $\displaystyle\int_{-2}^{2} x^4 - 4x^2 + 6 \, dx.$
3, p. 349

Notice $(-x)^4 - 4(-x)^2 + 6 = x^4 - 4x^2 + 6$, so the integrand is an even function of x. Moreover, $[-2,2]$ is a symmetric interval, so by Theorem 8,

$$\int_{-2}^{2} x^4 - 4x^2 + 6 \, dx = 2\int_0^2 x^4 - 4x^2 + 6 \, dx$$

$$= 2\left(\frac{1}{5}x^5 - \frac{4}{3}x^3 + 6x \Big|_0^2 \right)$$

$$= 2\left(\frac{1}{5}(2)^5 - \frac{4}{3}(2)^3 + 6(2) - \frac{1}{5}(0)^5 + \frac{4}{3}(0)^3 + 6(0) \right)$$

$$= 2\left(\frac{1}{5}(32) - \frac{4}{3}(8) + 12 \right)$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12 \right)$$

$$= \frac{2(32(3) - 32(5) + 12(15))}{15}$$

$$= \frac{2(32(-2) + 12(15))}{15}$$

$$= \frac{2(-64 + 180)}{15}$$

$$= \frac{2(116)}{15} = \frac{232}{15} \approx \boxed{15.5}$$

Area between curves.

If f and g are continuous <ins>and</ins> $f(x) \geq g(x)$ for all $x \in [a,b]$,

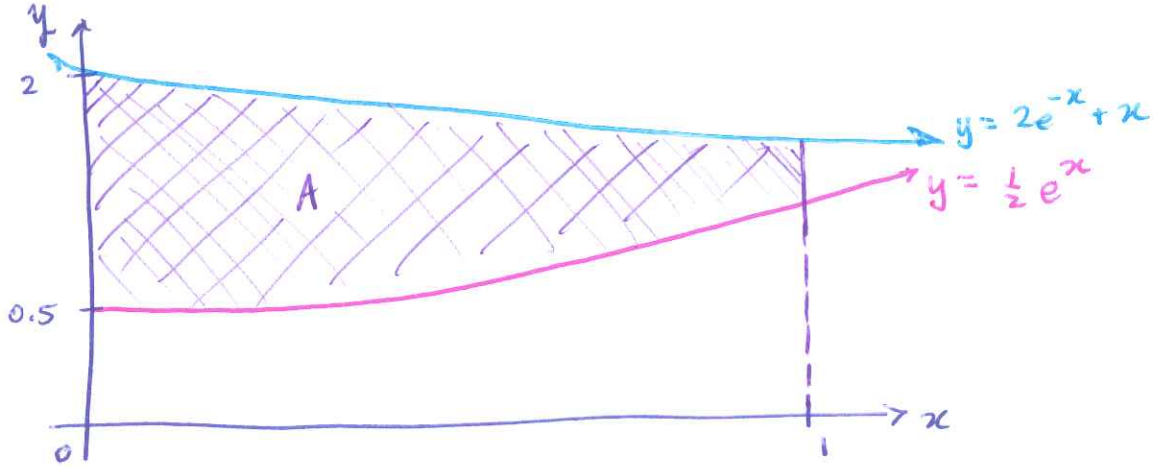
Then the area of the region between the curves $y = f(x)$ and $y = g(x)$ is:

$$A = \int_a^b f(x) - g(x) \, dx .$$

<ins>NOTE</ins>:
- It helps to graph the curves, if you don't know which one is on top
- Sometimes you need to find points of intersection algebraically.

Example 4, p. 350

Find the area of the region bounded above by $y = 2e^{-x} + x$, below by $y = \frac{e^x}{2}$, to the left by $x = 0$, and to the right by $x = 1$.



$$A = \int_0^1 2e^{-x} + x - \tfrac{1}{2}e^x \, dx$$

$$= -2e^{-x} + \tfrac{1}{2}x^2 - \tfrac{1}{2}e^x \Big|_0^1$$

$$= -2e^{-1} + \tfrac{1}{2}(1)^2 - \tfrac{1}{2}e^1 + 2e^{-0} - \tfrac{1}{2}(0)^2 + \tfrac{1}{2}e^0$$

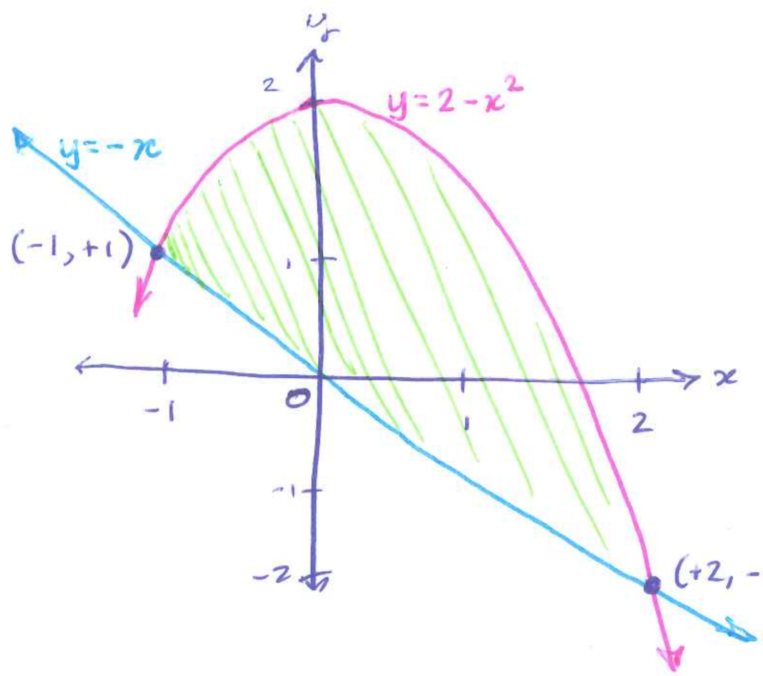
$$= -\frac{2}{e} + \frac{1}{2} - \frac{e}{2} + 2 + \frac{1}{2}$$

$$= 3 - \frac{2}{e} - \frac{e}{2}$$

$$= \frac{6e - 4 - e^2}{2e} \approx \boxed{0.9051}$$

Example 5, p. 350 — Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

① Sketch the curves



② Find points of intersection

Set $2 - x^2 := -x$

$\Leftrightarrow \quad x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

so $2 - x^2 = -x$ when $x = 2$ and when $x = -1$.

Example ct'd.
5, p. 350

(3) Which curve is on top?

$$y = 2 - x^2 \geq -x \quad \text{for } x \in [-1, 2].$$

(4) Set up integral

$$A = \int_{-1}^{2} (2 - x^2) - (-x)\, dx$$

(5) Integrate

$$= \int_{-1}^{2} 2 - x^2 + x \, dx$$

$$= 2x - \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_{-1}^{2}$$

$$= 2(2) - \frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 - 2(-1) + \frac{1}{3}(-1)^3 - \frac{1}{2}(-1)^2$$

$$= 4 - \frac{8}{3} + \frac{4}{2} + 2 - \frac{1}{3} - \frac{1}{2}$$

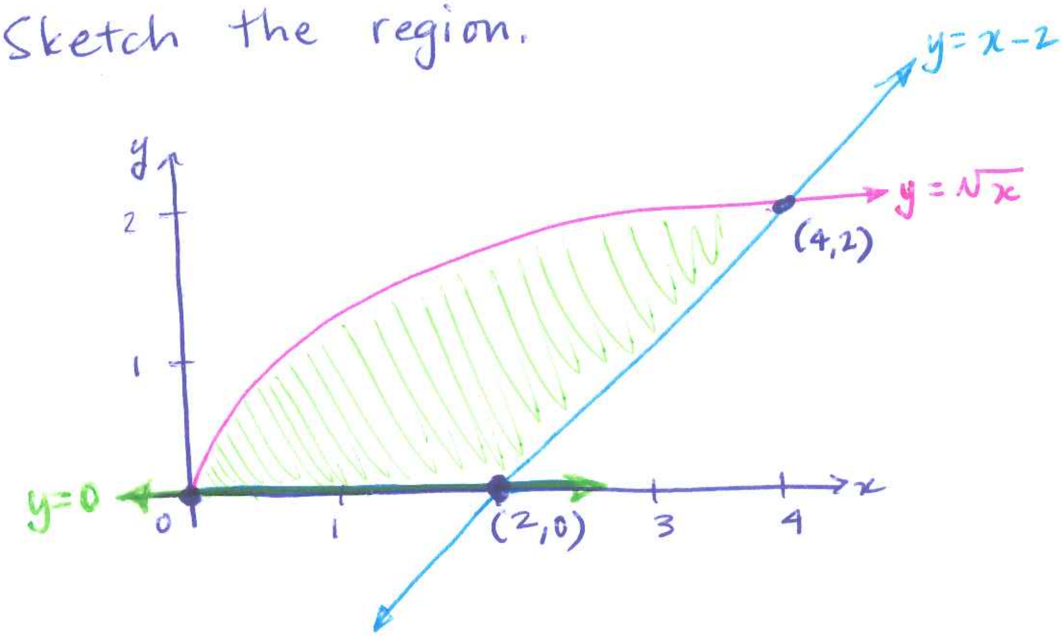
$$= 4 - \frac{8}{3} + 2 + 2 - \frac{1}{3} - \frac{1}{2}$$

$$= 4 - \frac{9}{3} - \frac{1}{2} + 4$$

$$= 8 - 3 - \frac{1}{2} = \boxed{5\tfrac{1}{2}}$$

Example 6, p. 351 Find the area of the region in the first quadrant bounded above by $y=\sqrt{x}$ and below by the x-axis and the line $y=x-2$.

① Sketch the region.



② Compute points of intersection.

- Set $\sqrt{x} :=: x-2$ $\iff$ $x = (x-2)^2$, $\sqrt{x} \geq 0$
  to find int. of $y=\sqrt{x}$ with $y=x-2$
  $\iff$ $x = x^2-4x+4$, $\sqrt{x} \geq 0$
  $\iff$ $x^2-5x+4=0$, $\sqrt{x} \geq 0$
  $\iff$ $(x-4)(x-1)=0$
  $x=4$; $\cancel{x=1}$ gives $y=x-2 \geq 0$.
- Set $\sqrt{x} :=: 0$ $\iff$ $x=0$
  to find intersection of $y=\sqrt{x}$ with $y=0$
- Set $x-2 :=: 0$ $\iff$ $x=2$
  to find intersection of $y=x-2$ with $y=0$.

(3) check which function is on top of which other:

- For $x \in [0,2]$, $y = \sqrt{x}$ is on top of $y = 0$
- For $x \in [2,4]$, $y = \sqrt{x}$ is on top of $y = x-2$.

(4) Set up integrals:

$$A = \int_0^2 \sqrt{x} - 0 \, dx + \int_2^4 \sqrt{x} - (x-2) \, dx$$

(5)

$$= \int_0^2 x^{1/2} \, dx + \int_2^4 x^{1/2} - x + 2 \, dx$$

$$= \left[\frac{2}{3} x^{3/2} \Big|_0^2 \right] + \left[\frac{2}{3} x^{3/2} - \frac{1}{2} x^2 + 2x \Big|_2^4 \right]$$

$$= \left[\frac{2}{3}(2)^{3/2} - \frac{2}{3}(0)^{3/2} \right] + \left[\frac{2}{3}(4)^{3/2} - \frac{1}{2}(4)^2 + 2(4) - \right.$$
$$\left. - \frac{2}{3}(2)^{3/2} + \frac{1}{2}(2)^2 - 2(2) \right]$$

$$= \frac{2}{3}(2)^3 - \frac{1}{2}(16) + 8 + \frac{1}{2}(4) - 4$$

$$= \frac{16}{3} - 2 = \frac{16-6}{2} = \frac{10}{3} \approx \boxed{3.33}$$