Lecture 5 : May 31, 2016.

Announcements / Assignments.

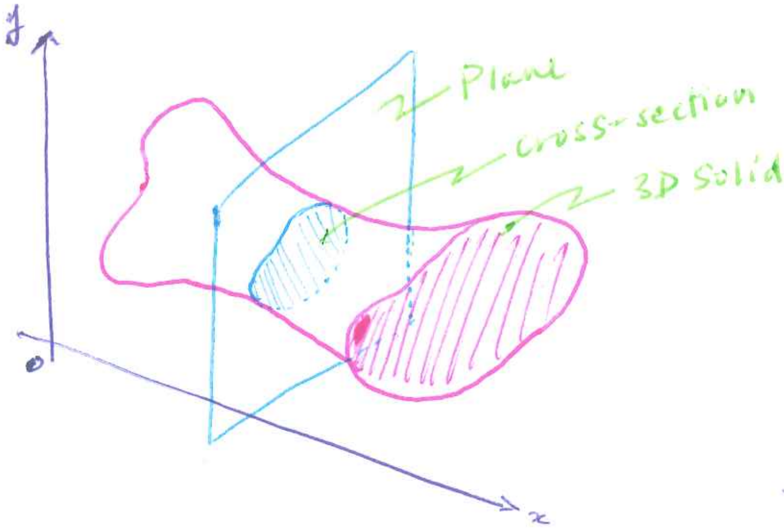
- Webwork 5 due Friday night
- ~~[illegible]~~ Homework 2 due Friday night – last week's material
- Homework 3 due Monday night (to be posted) – this wk. material
- Midterm exam ONE WEEK from today – see syllabus.

– Questions? –

TODAY :

- 6.1 : Volumes using cross-sections
- 6.3 : Arc length

6.1: Volumes using cross-sections.



DEF. A cross-section of a solid is the region formed by intersecting that solid with a plane.

... how could we use this to compute volume?

Recall: For a cylindrical solid with known base, the volume is:
$$\begin{pmatrix}\text{volume}\\ \text{of cylinder}\end{pmatrix} = \begin{pmatrix}\text{Area of}\\ \text{base}\end{pmatrix} \times \begin{pmatrix}\text{height of}\\ \text{cylinder}\end{pmatrix}$$

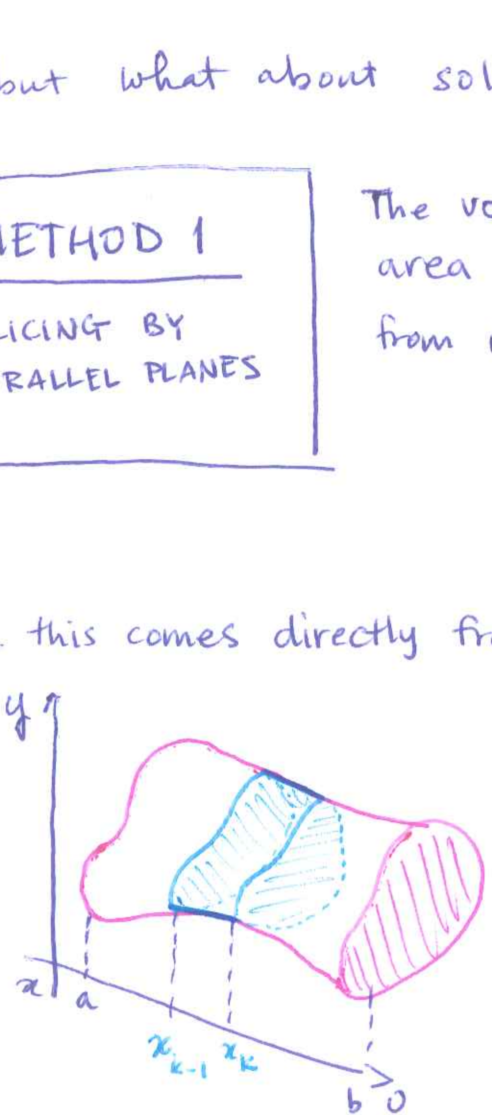
... but what about solids that are not cylinders?

METHOD 1

SLICING BY PARALLEL PLANES

The volume of a solid whose cross-sectional area is the integrable function $A(x)$, from $x = a$ to $x = b$, is
$$V = \int_a^b A(x)\,dx.$$

... this comes directly from a "Riemann-sum-like" slicing:



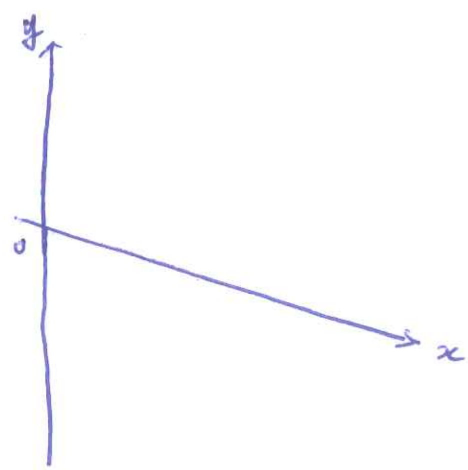
So, to compute volume:

1. Sketch the solid AND sketch a typical cross-section
2. Find a formula for $A(x)$, the area of a typical cross-section
3. Find the limits of integration
4. Integrate $A(x)$ to find the volume.

Example 1, p. 366

A pyramid 3m high has a square base that is 3m on a side. The cross-section of the pyramid perpendicular to the altitude, x meters down from the vertex, is a square x meters on each side. Find the volume of the pyramid.

Sol: 1. Sketch



Example 1 ct'd.

2. Find a formula for $A(x)$.

[We know that the cross-section is a square - see problem description.]

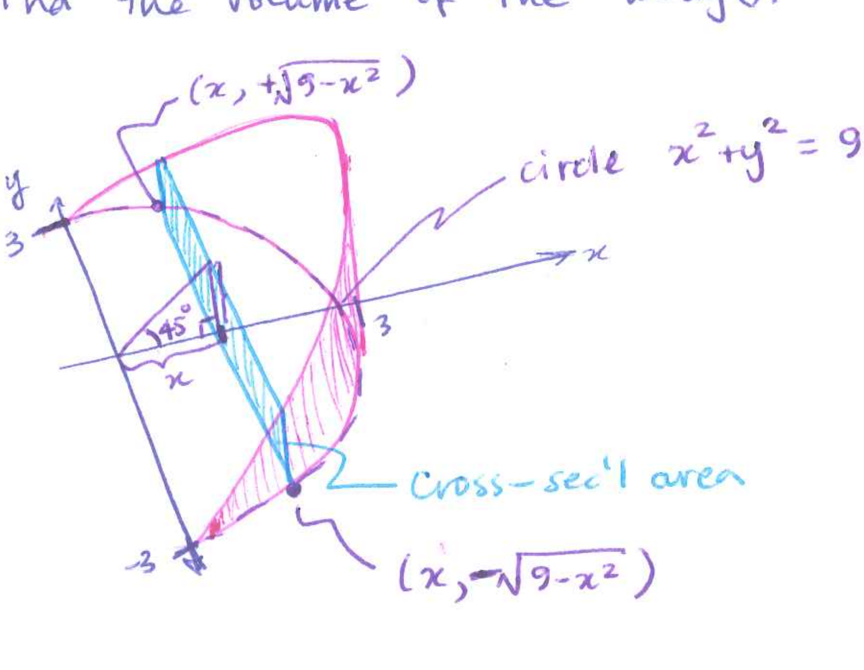
3. Find limits of integration.

4. Integrate.

Example 2, p.367

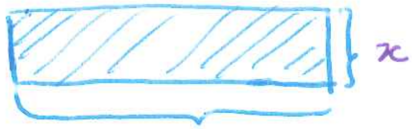
A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

1. Sketch.



Example 2 ct'd

2. Find a formula for $A(x)$.



$$\sqrt{9-x^2} - \left(-\sqrt{9-x^2}\right) = 2\sqrt{9-x^2}$$

So $A(x) = 2x\sqrt{9-x^2}$.

3. Limits of integration.

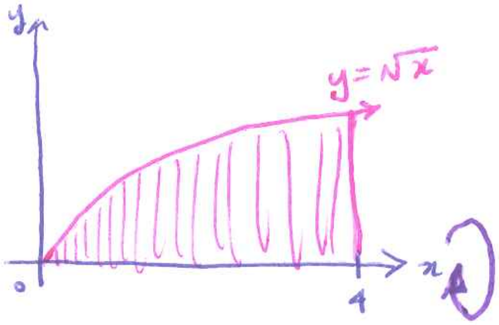
From $x=0$ to $x=3$. (Visual)

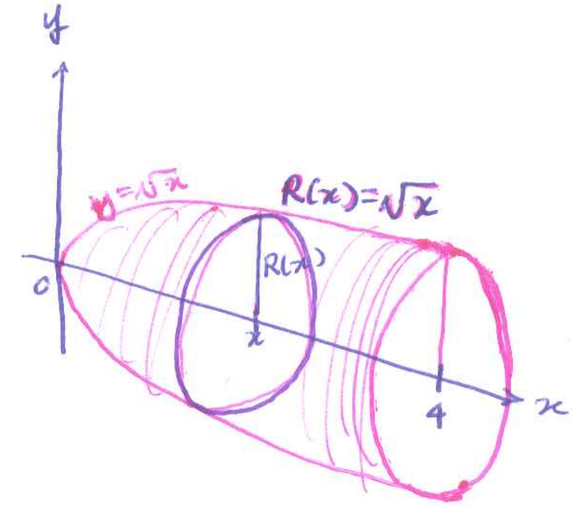
4. Integrate:

$$V = \int_0^3 2x\sqrt{9-x^2}\,dx.$$

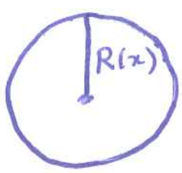
For <u>SOLIDS OF REVOLUTION</u>, the method of slicing into parallel planes has a special form.

... a solid of revolution is a special kind of solid formed by rotating a plane region about an axis in its plane:





In this case, the cross-sectional area is:



$A =$ ________

And so the volume formula becomes

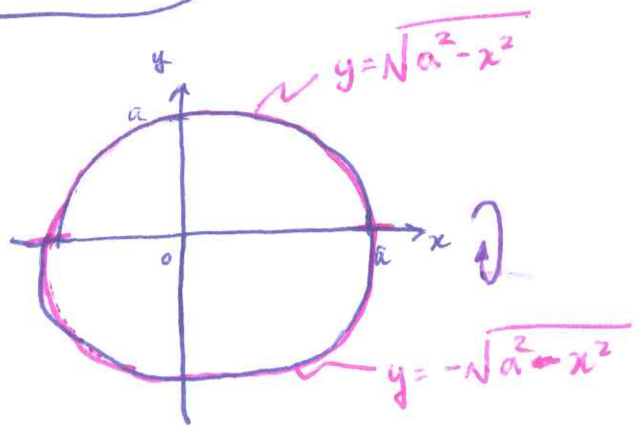
$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

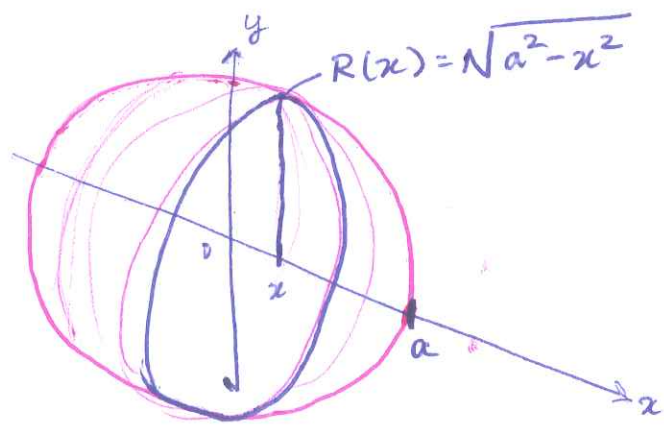
Example 4, p.368 Find the volume of the solid shown above, generated by rotating the region between the graph of $y = \sqrt{x}$ and the x-axis for $0 \le x \le 4$, about the x-axis.

Sol. $V = \int_a^b \pi [R(x)]^2 \, dx = \int_0^4 \pi (\sqrt{x})^2 \, dx$

$=$

Example 5, p.368 The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.



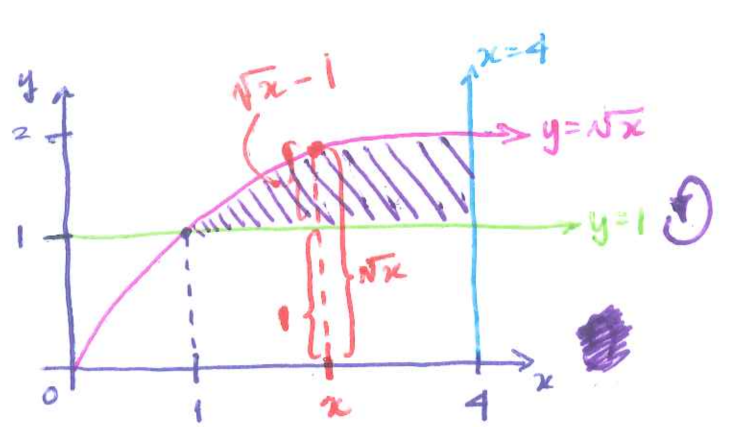


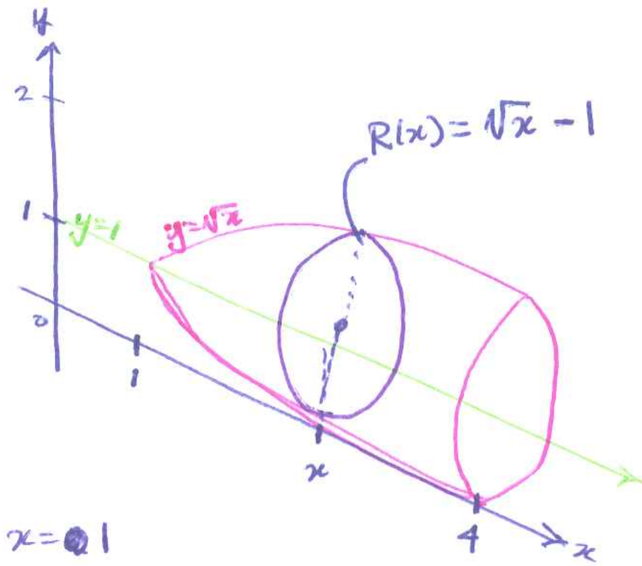
Limits: $0 \le x \le a$.

So $V = \int_0^a \pi \left(\sqrt{a^2 - x^2}\right)^2 dx = \int_0^a \pi(a^2 - x^2) \, dx = \int_0^a \pi a^2 - \pi x^2 \, dx$

Example 6, p. 368

Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$.





So $R(x)=\sqrt{x}-1$, with limits $x=1$ and $x=4$.

Then
$$V=\int_1^4 \pi(\sqrt{x}-1)^2\,dx = \int_1^4 \pi(x-2\sqrt{x}+1)\,dx$$

$$= \int_1^4 \pi x - 2\pi x^{1/2} + \pi \,dx$$

$$= \pi\frac{x^2}{2} - \frac{2\pi x^{3/2}}{3/2} + \pi x \Big|_1^4$$

$$= \frac{\pi}{2}x^2 - \frac{4\pi}{3}x^{3/2} + \pi x \Big|_1^4$$

$$= \frac{\pi}{2}(4)^2 - \frac{4\pi}{3}(4)^{3/2} + 4\pi - \frac{\pi}{2}(1)^2 + \frac{4\pi}{3}(1)^{3/2} - \pi(1)$$

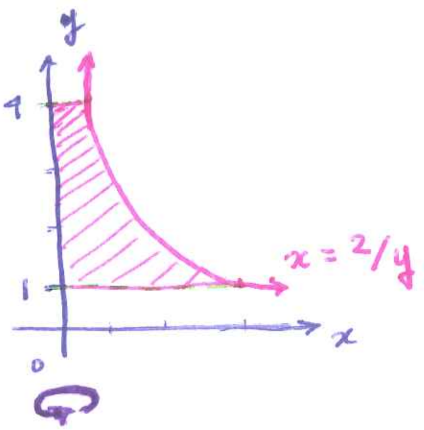
$$= \frac{\pi}{2}(16) - \frac{4\pi}{3}(2)^3 + 4\pi - \frac{\pi}{2} + \frac{4\pi}{3} - \pi$$

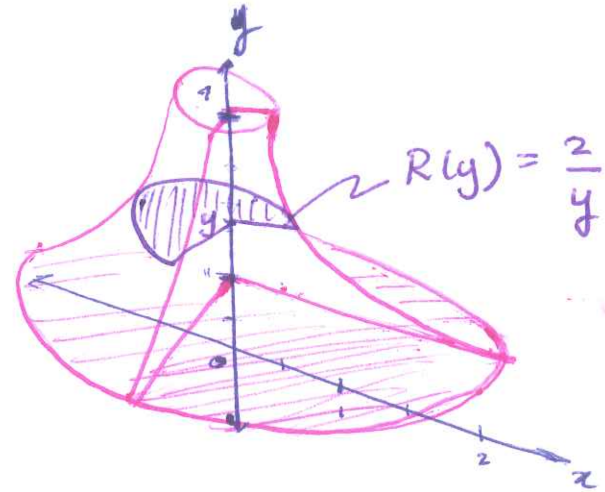
$$= 8\pi - \frac{4\cdot 8\pi}{3} + 4\pi - \frac{\pi}{2} + \frac{4\pi}{3} - \pi = 11\pi - \frac{28\pi}{3} - \frac{\pi}{2} = \frac{7\pi}{6}.$$

Similar method for curves in y-variable rotated about the y-axis:

Example 7, p. 370 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$, about the y-axis.

Sol.





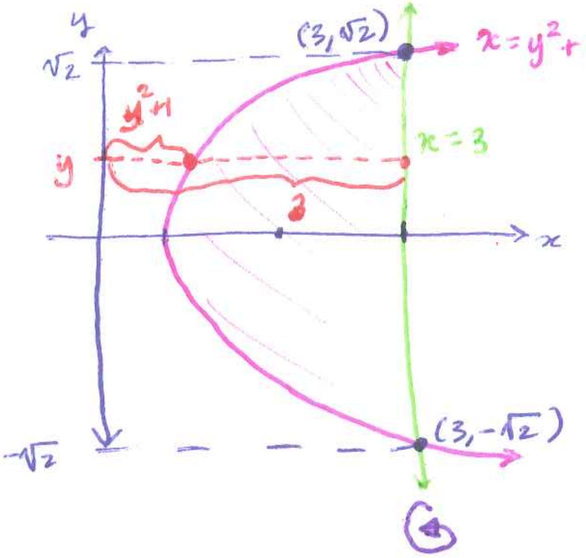
So $R(y) = \frac{2}{y}$, limits are $y = 1$ and $y = 4$. So:

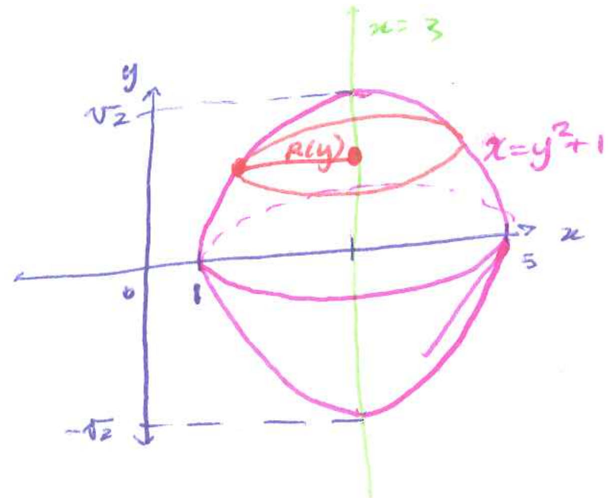
$$V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = \int_1^4 \frac{4\pi}{y^2}\, dy = \int_1^4 4\pi\, y^{-2}\, dy$$

$$= \frac{4\pi}{-1} y^{-1} \Big|_1^4 = -\frac{4\pi}{y} \Big|_1^4$$

$$= -\frac{4\pi}{4} + \frac{4\pi}{1} = -\pi + 4\pi = 3\pi.$$

Example 8, p. 370 Find the volume of the solid generated by revolving the region between the parabola $x=y^2+1$ and the line $x=3$ about the line $x=3$.





So $R(y) = 3-(y^2+1) = 2-y^2$ and limits: $y=-\sqrt{2}$ to $y=\sqrt{2}$.

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi (2-y^2)^2\, dy$$

NOTE: $f(y) := \pi(2-y^2)^2$ has $f(-y) = \pi(2-(-y)^2)^2 = \pi(2-y^2)^2$, so it is an even function, and $[-\sqrt{2},\sqrt{2}]$ is a symmetric interval!

$$= 2\int_0^{\sqrt{2}} \pi (2-y^2)^2\, dy$$

$$= 2\pi \int_0^{\pi} 4-4y^2+y^4\, dy$$

$$= 2\pi \left[4y - \frac{4}{3}y^3 + \frac{1}{5}y^5\right]_0^{\sqrt{2}}$$

$$= 2\pi\left[4\sqrt{2} - \frac{4}{3}(\sqrt{2})^3 + \frac{1}{5}(\sqrt{2})^5\right] - 2\pi\left[4(0) - \frac{4}{3}(0)^3 + \frac{1}{5}(0)^5\right]$$

$$= 2\pi\left[4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2}\right] = 2\pi\left[\frac{4\cdot 15 - 8\cdot 5 + 4\cdot 3}{15}\sqrt{2}\right] = \frac{64\pi\sqrt{2}}{15}.$$

"Washer method" for solids of revolution — useful when the axis of revolution does not lie on the region revolved:

... a logical extension of the disk method:

$$V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2 \right) dx,$$

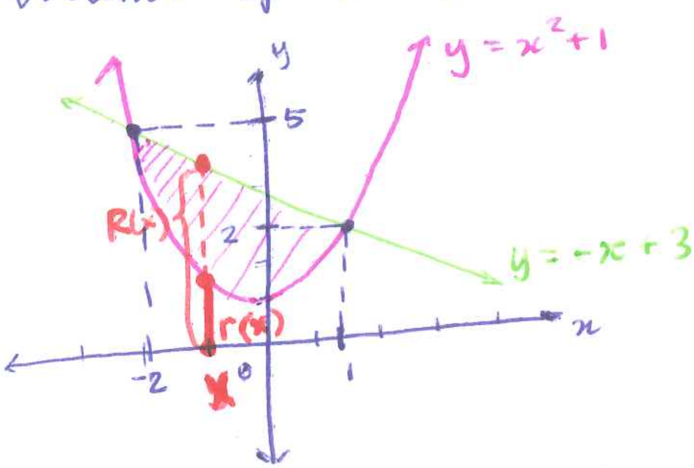
where $R(x)$: outer radius, $r(x)$: inner radius.

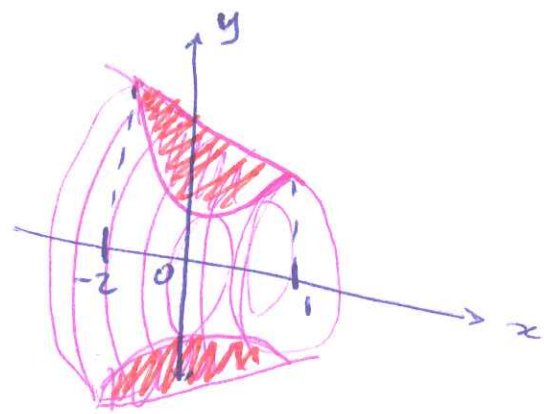
Note: For each of these problems, still follow steps:

1. Sketch
2. Find radius (inner & outer) or a formula for cross-sectional area
3. Find limits of integration
4. Integrate.

Example 9, p. 371

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.





1. Sketches

2. Outer radius: $R(x) = -x + 3$

   Inner radius: $r(x) = x^2 + 1$

3. Limits: $x = -2$ to $x = 1$

4. Integrate:

$$V = \int_{-2}^{1} \pi \left[(-x+3)^2 - (x^2+1)^2 \right] dx$$

$$= \int_{-2}^{1} \pi \left[x^2 - 6x + 9 - x^4 - 2x^2 - 1 \right] dx$$

$$= \int_{-2}^{1} \pi \left[-x^4 - x^2 - 6x + 8 \right] dx$$

Example 9, ct'd

$$V = \int_{-2}^{1} \pi \left[-x^4 - x^2 - 6x + 8\right] dx$$

$$= \pi \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{6}{2}x^2 + 8x\right]_{-2}^{1}$$

$$= \pi \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 - 3x^2 + 8x\right]_{-2}^{1}$$

$$= \pi \left[-\frac{1}{5}(1)^5 - \frac{1}{3}(1)^3 - 3(1)^3 + 8(1)\right] -$$

$$- \pi \left[-\frac{1}{5}(-2)^5 - \frac{1}{3}(-2)^3 - 3(-2)^2 + 8(-2)\right]$$

$$= \pi \left[-\frac{1}{5} - \frac{1}{3} - 3 + 8\right] - \pi \left[-\frac{1}{5}(-32) - \frac{1}{3}(-8) - 3(4) - 16\right]$$

$$= \pi \left[\frac{-3 - 5 + 5\cdot 15}{15}\right] - \pi \left[\frac{32\cdot 3 + 8\cdot 5 - 15\cdot 28}{15}\right]$$

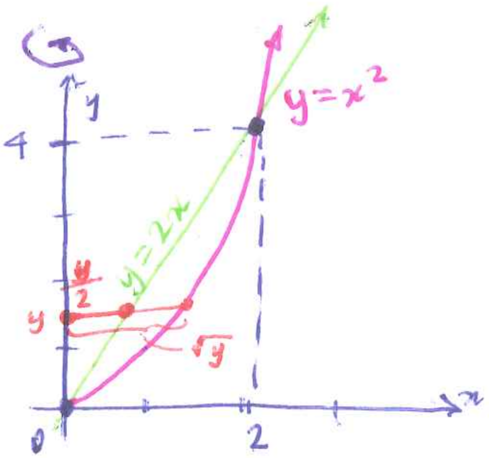
$$= \pi \left[\frac{15\cdot 5 - 8 - 32\cdot 3 - 40 + 15\cdot 28}{15}\right]$$

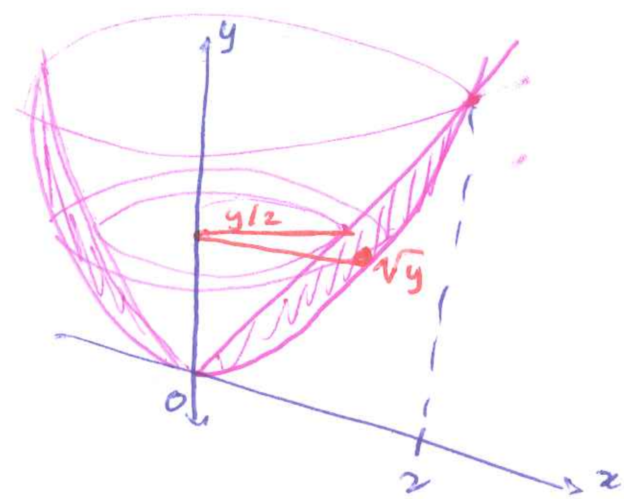
$$= \pi \left[\frac{15\cdot 32 - 32\cdot 3 - 48}{15}\right]$$

$$= \pi \left[\frac{32\cdot 12 - 48}{15}\right] = \pi \left[\frac{32\cdot 4 - 16}{5}\right] = \pi \left[\frac{117}{5}\right].$$

Example 10, p. 372

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of this solid.





So inner: $r(y) = y/2$

outer: $R(y) = \sqrt{y}$. Limits: $y = 0$, $y = 4$.

$$V = \int_0^4 \pi \left[(\sqrt{y})^2 - (y/2)^2 \right] dy = \int_0^4 \pi \left[y - \frac{y^2}{4} \right] dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{3 \cdot 4} \right]_0^4 = \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4$$

$$= \pi \left[\frac{4^2}{2} - \frac{4^3}{12} \right] - \pi \left[\frac{0^2}{2} - \frac{0^3}{12} \right]$$

$$= \pi \left[\frac{16}{2} - \frac{16}{3} \right]$$

$$= \pi \left[8 - \frac{16}{3} \right] = \pi \left[\frac{8 \cdot 3 - 8 \cdot 2}{3} \right] = \boxed{\pi \left[\frac{8}{3} \right]}.$$

6.3: Arc length.

DEFINITION. If $f'(x)$ is continuous on $[a,b]$, then the arc length of the curve $y=f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is:

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

This is a definition - so it doesn't need proof, but a derivation / justification is given on p. 384, and boils down to a Riemann sum of segment lengths.

Example 1, p. 385 Find the length of $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$, $0 \le x \le 1$.

Sol. 1. Compute $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{4\sqrt{2}}{3}x^{3/2} - 1\right] = \frac{4\sqrt{2}}{3}\left(\frac{3}{2}\right)x^{1/2}$$

$$= \frac{12\sqrt{2}}{6}\sqrt{x} = 2\sqrt{2x}.$$

2. Compute $\left(\frac{dy}{dx}\right)^2$.

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2x}\right)^2 = 4(2x) = 8x.$$

Example 1, ct'd

3. Integrate over limits:

$$L = \int_0^1 \sqrt{1 + (dy/dx)^2} \, dx$$

$$= \int_0^1 \sqrt{1 + 8x} \, dx .$$

Let $u := 1 + 8x$, $du = \frac{du}{dx} dx = 8\,dx \Leftrightarrow dx = \frac{1}{8} du$

$u(0) = 1 + 8(0) = 1$
$u(1) = 1 + 8(1) = 9$

Then $L = \int_1^9 \sqrt{u} \left(\frac{1}{8} du \right) = \frac{1}{8} \int_1^9 \sqrt{u} \, du$

$$= \frac{1}{8} \left(\frac{2}{3} \right) u^{3/2} \Big|_1^9$$

$$= \frac{1}{4 \cdot 3} u^{3/2} \Big|_1^9 = \frac{1}{12} u^{3/2} \Big|_1^9$$

$$= \frac{1}{12} (9)^{3/2} - \frac{1}{12} (1)^{3/2}$$

$$= \frac{1}{12} (3)^3 - \frac{1}{12}$$

$$= \frac{27 - 1}{12} = \frac{26}{12} = \frac{13}{6} \approx \boxed{2.17.}$$

Example 2, p. 385

Find the length of the graph of $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \le x \le 4$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3}{12} + \frac{1}{x}\right] = \frac{3x^2}{12} - \frac{1}{x^2} = \frac{x^2}{4} - \frac{1}{x^2} =$$

$$= \frac{x^4 - 4}{4x^2}.$$

So $$\left(\frac{dy}{dx}\right)^2 = \left(\frac{x^4-4}{4x^2}\right)^2 = \frac{x^8 - 8x^4 + 16}{16x^4}.$$

Then the arc length is:

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{x^8 - 8x^4 + 16}{16x^4}}\, dx$$

$$= \int_1^4 \sqrt{\frac{16x^4 + x^8 - 8x^4 + 16}{16x^4}}\, dx = \int_1^4 \sqrt{\frac{x^8 + 8x^4 + 16}{16x^4}}\, dx$$

$$= \int_1^4 \sqrt{\frac{(x^4+4)^2}{(4x^2)^2}}\, dx = \int_1^4 \frac{x^4+4}{4x^2}\, dx$$

$$= \int_1^4 \frac{x^2}{4} + \frac{1}{x^2}\, dx = \frac{x^3}{12} - \frac{1}{x}\Big|_1^4 =$$

$$= \frac{4^3}{12} - \frac{1}{4} - \frac{1^3}{12} + \frac{1}{1} = \frac{16}{3} - \frac{1}{4} - \frac{1}{12} + 1 = \frac{16 \cdot 4 - 3 - 1 + 12}{12} = \frac{72}{12} = \boxed{6}$$

Example 3, p. 386

Find the length of $y=\frac{1}{2}(e^{x}+e^{-x}),\ 0\le x\le 2.$

1. Find $\frac{dy}{dp}$. $\quad \frac{dy}{du}=\frac{d}{dp}\left[\frac{1}{2}(e^{x}+e^{-x})\right]=\frac{1}{2}(e^{-x}-e^{-x}).$

2. Find $\left(\frac{dy}{dx}\right)^2$. $\quad \left(\frac{dy}{dx}\right)^2=\left(\frac{1}{2}(e^{-x}-e^{-x})\right)^2$

$$=\frac{1}{4}\left(e^{-2x}-2e^{0}+e^{2x}\right)$$

$$=\frac{1}{4}\left(e^{-2x}-2+e^{2x}\right).$$

3. Find $1+\left(\frac{dy}{dx}\right)^2$. $\quad 1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4}\left(e^{-2x}-2+e^{2x}\right)$

$$=\frac{1}{4}\left(e^{-2x}+2+e^{2x}\right)$$

$$=\frac{1}{4}\left(e^{x}+e^{-x}\right)^2.$$

So $L=\int_0^2\sqrt{\frac{1}{4}(e^{x}+e^{-x})^2}\,dx=\int_0^2\frac{1}{2}(e^{x}+e^{-x})\,dx$

$$=\frac{1}{2}e^{x}-\frac{1}{2}e^{-x}\Big|_0^2=\frac{1}{2}e^{2}-\frac{1}{2}e^{-2}-\frac{1}{2}e^{0}+\frac{1}{2}e^{-0}$$

$$=\frac{1}{2}e^{2}-\frac{1}{2e^{2}}-\frac{1}{2}+\frac{1}{2}=\frac{e^{2}}{2}-\frac{1}{2e^{2}}\approx 3.63$$

Example 9, p. 387 Find the length of $y = \left(\frac{x}{2}\right)^{2/3}$ from $x=0$ to $x=2$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right)^{2/3} = \frac{d}{dx}\left[\left(\tfrac{1}{2}\right)^{2/3} x^{2/3}\right] = \left(\tfrac{1}{2}\right)^{2/3}\left(\tfrac{2}{3}\right)x^{-1/3} = \frac{1}{3}\left(\frac{x}{2}\right)^{-1/3}$$

NOTE !! The derivative $\frac{1}{3}\left(\frac{x}{2}\right)^{-1/3} = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$ is

NOT DEFINED at $x=0$!! Problem !

However, redefine the curve – variable in y works:

$$y = \left(\frac{x}{2}\right)^{2/3} \iff y^{3/2} = \frac{x}{2} \iff x = 2y^{3/2}.$$

Then $\frac{dx}{dy} = \frac{d}{dy}\left[2y^{3/2}\right] = 2\left(\frac{3}{2}\right)y^{1/2} = 3\sqrt{y}$.

Limits are $x=0 \Rightarrow y = \left(\frac{0}{2}\right)^{2/3} = 0$

$x = 2 \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} = 1.$

And $\frac{dx}{dy}$ is defined for $y \in [0,1]$.

So $\left(\frac{dx}{dy}\right)^2 = (3\sqrt{y})^2 = 9y$, and therefore

$$L = \int_0^1 \sqrt{1+9y}\,dy = \frac{1}{9}\int_1^{10} \sqrt{u}\,du = \frac{2}{9\cdot 3}u^{3/2}\Big|_1^{10}$$

$u = 1+9y \quad u(0)=1$
$du = 9\,dy \quad u(1)=10$

$$= \frac{2}{9\cdot 3}(10)^{3/2} - \frac{2}{9\cdot 3}(1)^{3/2}$$

$$= \frac{2}{27}\left(10\sqrt{10}-1\right) \approx 2.27$$