Lecture 7: June 9.

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Announcements/ Assignments.

- HW 2/3 grades to be posted this week
- Midterm grades posted

  Range: 57 – 81 out of 90 (that's 63 – 90/100)

  Mean: 68.2 out of 90 (that's 76/100)

  Standard deviation: 8/90 (or 9/100)

- No written HW due Monday (you did HW3 last week)
- There is WW due Monday on tonight's material

Tonight:

- 7.1: the Natural Logarithm as an Integral
- Exam Review/Recap?

7.1 : Natural Logarithm as Integral

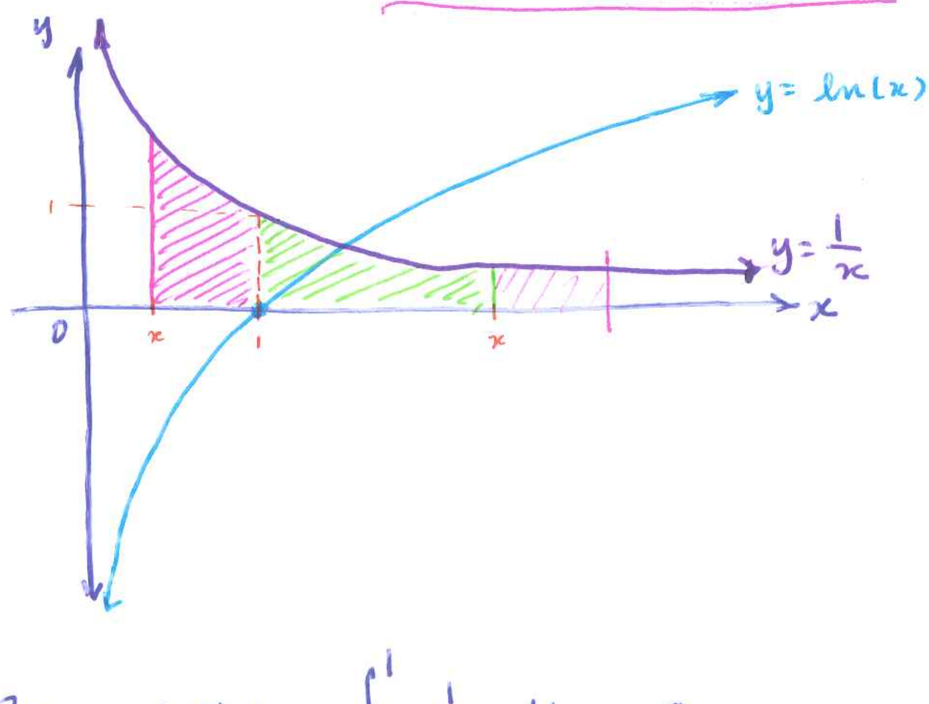
$$\boxed{\underline{\text{DEF.}} \quad \text{For } x>0, \quad \ln(x) := \int_1^x \frac{1}{t}\, dt}$$

A <u>definition</u> — not a <u>result</u>. Let's see what this means.

- The FTC says that the integral is a ~~a~~ continuous function. This means $\ln(x)$ is continuous.

- Swapping bounds:

For $0 < x < 1$, $\underline{\int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt}$.



- Zero-width: $\int_1^1 \frac{1}{t}\, dt = 0$

- Note the difference between x and t:

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \quad , \text{ NOT } \cancel{\int_1^x \frac{1}{x}\,dx}$$

- How to numerically approximate $\ln(x)$?

... compute Riemann sums.

When you do this:

(to two decimal places)

x	0	0.05	0.5	1	2
$\ln(x)$	undef.	-3.00	-0.69	0	0.69

x	3	4	10	... etc.
$\ln(x)$	1.10	1.39	2.30	

Q. For which x does $\ln(x) = 1$?

Does there exist such an x?

Recall: the Intermediate Value Theorem.

> **THM p. 99:** If f is a continuous fn. on a closed interval $[a,b]$, and $f(a) \le y_0 \le f(b)$,
> Then there exists $c \in [a,b]$ s.t. $f(c) = y_0$.

... what about $\ln(x)$?

- Continuous? ✓
- Closed interval? $[2,3]$

  $f(2) \approx 0.69$, $f(3) \approx 1.10$ $\quad y_0 = 1$

... So for some $c \in [2,3]$, $\ln(c) = 1$.

* $\boxed{\text{Call that number "e"}}$ *

> **DEF.** The number e is the one in the domain of the natural logarithm satisfying
>
> $$\ln(e) = \underbrace{\int_1^e \frac{1}{t}\,dt}_{\text{defn of } \ln(e)} = 1.$$

(... yes, this is the same "e" we know/love)

Q. How to compute $\frac{d}{dx}\left[\ln(x)\right]$?

$$\frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

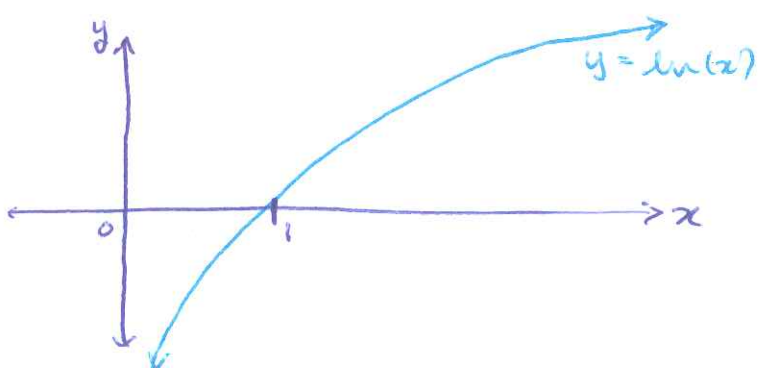
$$\frac{d^2}{dx^2}\left[\ln(x)\right] = \frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = \frac{-1}{x^2}.$$

Q. How to graph $\ln(x)$?

What is the range?

Range: $x > 0$.

Graph:
- 1^{st} deriv. tells us that $\ln(x)$ is always increasing. (because $\frac{1}{x} > 0$ for $x > 0$)
- 2^{nd} deriv. tells us that $\ln(x)$ is always concave down ⌒ (because $\frac{-1}{x^2} < 0$ for all x)
- Knew $\ln(1) = 0$ (because $\int_1^1 \frac{1}{t}\,dt = 0$)



- Properties of the natural logarithm

1. $\ln(bx) = \ln(b) + \ln(x)$, $\forall b > 0$

2. $\ln\left(\frac{b}{x}\right) = \ln(b) - \ln(x)$

3. $\ln\left(\frac{1}{x}\right) = -\ln(x)$

4. $\ln(x^r) = r \cdot \ln(x)$, $r \in \mathbb{Q}$.

$\mathbb{Q}$ is the set of rational #'s

i.e., $\frac{a}{b}$ where a, b - integers

- What is $\lim_{x \to \infty} \ln x$?

Easier (?): What is $\lim_{x\to\infty} \ln(2^x) \stackrel{?}{=} \infty$

Well, $\lim_{n \to \infty} \ln(2^n)$, $n \in \mathbb{N}$

$= \lim_{n\to\infty} (n \cdot \ln(2)) \approx \lim_{n\to\infty} 0.69\, n$

$= \infty$.

This tells us that $\lim_{x \to \infty} \ln(x) = \infty$ also.

(This is because for all $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $2^n \le x$ and $2^{n+1} > x$.)

- Integral of $\int \frac{1}{u}\,du$

* IF u is a diff'ble function tht. is never 0,

* THEN $\int \frac{1}{u}\,du = \ln|u| + c$

Example.

$$\int_{-\pi/2}^{\pi/2} \underbrace{\frac{4\cos\theta}{3+2\sin\theta}}_{:=f(\theta)}\,d\theta$$

$$f(-\theta) = \frac{4\cos(-\theta)}{3+2\sin(-\theta)} = \frac{4\cos(\theta)}{3-2\sin(\theta)} \neq \frac{\pm 4\cos\theta}{3+2\sin\theta}$$

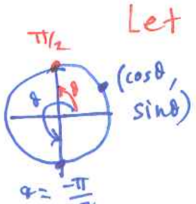
So, this is neither odd nor even; just compute:

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta = \int_1^5 \frac{2\,du}{u} = 2\int_1^5 \frac{1}{u}\,du$$

Let $u := 3+2\sin\theta$. Then $du = 2\cos\theta$

$u(-\pi/2) = 3 + 2\sin(-\pi/2) = 3-2 = 1$

$u(\pi/2) = 3 + 2\sin(\pi/2) = 3+2 = 5$



$$= 2\ln|u|\Big|_1^5$$

$$= 2\left(\ln|5| - \underbrace{\ln|1|}_{=0}\right)$$

$$= \boxed{2\ln(5)}$$

- The inverse of $\ln(x)$ and the number e

Observe: For all a, $\ln(a^r) = r \cdot \ln(a)$.

Set $a = e$: $\ln(e^r) = r \cdot \underbrace{\ln(e)}_{=1 \text{ by def.}} = r$

$\ln(e^r) = r$ Apply $\exp(\cdot)$ to both sides

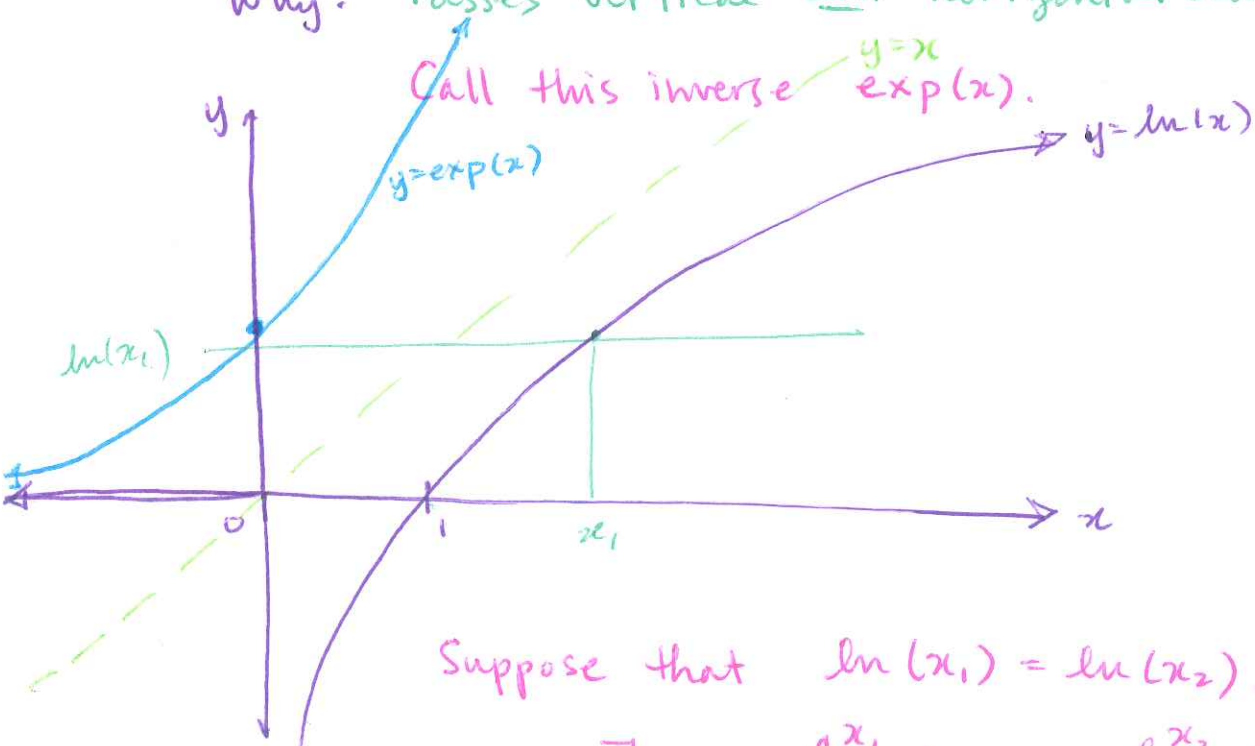
$\exp(\ln(e^r)) = \exp(r)$

$$\boxed{e^r = \exp(r)}$$

But also... we know $\ln(x)$ is invertible.

Why? Passes vertical and horizontal line test.

Call this inverse $\exp(x)$.



Suppose that $\ln(x_1) = \ln(x_2)$.

Then $\int_1^{x_1} \frac{1}{t}\,dt = \int_1^{x_2} \frac{1}{t}\,dt$.

So $\int_{x_1}^{x_2} \frac{1}{t}\,dt = 0$, but this is true only when $x_1 = x_2$.

Rules for exp, ln.

- $\exp(\ln(x)) = x$ for all $x > 0$
- $\ln(\exp(x)) = x$ for all x because $\underline{\exp(x) = e^x}$ and $e \in [2,3]$

- $\dfrac{d}{dx}\left[e^x\right] = e^x$ by Sn. 3.8 in text

  $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

- $\int e^x \, dx = e^x + c$

Laws of exponents.

① $e^a e^b = e^{a+b}$

② $\dfrac{e^a}{e^b} = e^{a-b}$

③ $e^{-a} = \dfrac{1}{e^a}$

④ $(e^a)^b = e^{ab} = (e^b)^a$

- General exp. fn.

$$\boxed{a^x := e^{x\ln(a)} \quad , \quad a > 0.}$$

- $$\frac{d}{dx}\left[a^x\right] = \frac{d}{dx}\left[e^{x\ln(a)}\right]$$
$$= \ln(a)\, e^{x\ln(a)}$$
$$= \ln(a)\, a^x.$$

- $$\int a^x \, dx = \frac{1}{\ln(a)} \cdot a^x + C$$

- General logarithmic fn:

$\log_a(x)$ is the inverse fn. of a^x.

- $\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{\ln(a)} \cdot \frac{1}{x}$