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LECTURE 8: June 14.

Announcements/assignments.

- Webwork 8 due Friday
- Homework 4 due Monday

Today.

7.2: Exponential change ; Separable differential eq'ns

8.1: Basic integration formulas.

7.1: Exponential change & Separable Diff. Eq'ns.

- **Exponential Change** is when a quantity increases or decreases at a rate proportional to the quantity's size at a given time.

Examples:
- Population growth
- Compounding interest * HW 4
- Radioactive decay
- Heat flow

In mathematics, if the quantity that is inc/dec. is called $y(t)$ then increasing/decreasing at a rate proportional to $y(t)$ is represented:

constant of proportionality

$$\frac{dy}{dt} = k\,y(t) = ky$$

If we say that the initial (starting) quantity is y_0, then we have the IVP:

initial value prob.

$$\begin{cases} dy/dt = ky \\ y(0) = y_0. \end{cases}$$

The IVP for exponential growth:

$$\begin{cases} \dfrac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}.$$

In week 1, we had

$$\begin{cases} \dfrac{dy}{dt} = f(t) \\ y(0) = y_0 \end{cases}.$$

A solution to this is $y(t) \equiv 0$.

Check: ① Eq'n: $\dfrac{dy}{dt} = \dfrac{d}{dt}[0] = 0 \overset{k\cdot 0}{''} = ky(t)$.

So $\dfrac{dy}{dt} = ky(t)$.

② $y(0) = 0$ (If $y_0 = 0$, solved IVP)

To find others, observe:

$\dfrac{dy}{dt} = ky$ — why not ~~$\int \frac{dy}{dt}\,dt = \int ky\,dt$, $y(t) = k\int y\,dt + C$~~

$\dfrac{1}{y}\dfrac{dy}{dt} = k$ — can do, because $y \neq 0$

$$\int_u \frac{1}{y} \boxed{\underbrace{\frac{dy}{dt}\,dt}_{du}} = \int k\,dt$$

Recall: $\int \frac{du}{u} = \ln|u| + C$

Let $u := y(t)$. Then $du = \frac{dy}{dt}\,dt$.

$\ln|y| + c_1 = kt + C_2$. But say $C := C_2 - C_1$

$\ln|y(t)| = kt + C$

$|y| = \exp(kt + c)$

$y = e^{kt+c} \iff y = Ae^{kt}$.

see pg 3(a)

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt$$

$u := y(t)$

$du = \frac{dy}{dt}\,dt$

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int \frac{du}{u} = \ln|u| + c_1 = \ln|y| + c_1$$

$$\int k\,dt = kt + c_2$$

$$\ln|y(t)| + c_1 = kt + c_2$$

$$\ln|y(t)| = kt + c_2 - c_1$$ Let $c := c_2 - c_1$

$$\ln|y(t)| = kt + c$$

$$|y(t)| = e^{kt+c}$$ $e^{kt+c} > 0$

$$y(t) = e^{kt+c}$$ $e^{a+b} = e^a e^b, \forall a,b.$

"for all"

$$y(t) = e^{kt}e^{c}$$ Let $A := e^c = e^{c_2-c_1}$

$$y(t) = Ae^{kt}$$

So $y(t) = Ae^{kt}$ solves $\frac{dy}{dt} = ky$.

To find A, use initial cond'n.

- by the $\overset{\text{initial condition}}{IC}$, $y(0) = y_0$
- from our solution, $y(0) = Ae^{0t} = Ae^0 = A$

Therefore, $y_0 = A$, and the solution is

$$y(t) = y_0 e^{kt}.$$

$$\boxed{\text{IMPORTANT: The sol'n of } \begin{cases} \frac{dy}{dt} = ky, \\ y(0) = y_0 \end{cases} \text{ is } y(t) = y_0 e^{kt}.}$$

If $k > 0$, y increases / ~~decreases~~ with t "exponential growth"

If $k < 0$, y ~~increases~~ / decreases with t. "exponential decay"

Recall: Rule of exponentials: $e^{ab} = (e^b)^a$, $\forall a, b$.

For us, means: $e^{kt} = (e^k)^t$. For $k > 0$, $e^k > 1$. $(e \approx 2.71\ldots)$

But for $k < 0$, $e^k = \frac{1}{e^{|k|}} < 1$.

Solving ODE's by separation.

A general first-order $\underset{\text{ordinary differential equation}}{\text{ODE}}$ is of the form

$$\frac{dy}{dx} = f(x,y)$$

and a solution to this ODE is a function $y = y(x)$ that satisfies $y'(x) = f(x, y(x))$.

> **DEF.**
> An ODE is called *separable* if it can be written:
>
> $$\frac{dy}{dx} = \underbrace{g(x)}_{\text{function of } x \text{ only}}\ \underbrace{H(y)}_{\text{function of } y \text{ only.}}$$
>
> e.g.
> - $\frac{dy}{dx} = \underbrace{(y-1)}_{H(y)}\overbrace{x^2}^{g(x)}$ is sep.
> - $\frac{dy}{dx} = y^x$ is not
> - $\frac{dy}{dx} = \cos(xy)$

Special technique for solving:

$$\frac{dy}{dx} = g(x)H(y) \iff \frac{dy}{dx} = \frac{g(x)}{h(y)} \iff h(y)\frac{dy}{dx} = g(x)$$

we define a new $h(y) := \frac{1}{H(y)}$

Then $\overbrace{\int h(y)\frac{dy}{dx}\,dx}^{= \int h(y)\,dy} = \int g(x)\,dx$. integrate w.r.t. x

Example 1, p. 432

Solve $\frac{dy}{dx} = (1+y)e^x$ for $y > -1$.

Form: $\frac{dy}{dx} = g(x)H(y)$, $g(x) = e^x$
$H(y) = 1+y$.

Then $\frac{1}{1+y}\frac{dy}{dx} = e^x$ (works bec. $1+y \neq 0$)

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int e^x\,dx$$

$$\int \frac{1}{1+y}\,dy = \int e^x\,dx$$

$$\int \frac{1}{1+y}\,dy = \int \frac{1}{u}\,du = \ln|u| + c_1 = \ln|1+y| + c_1.$$

Let $u := 1+y$
$du = dy$

$$\int e^x\,dx = e^x + c_2$$

Therefore, $\ln|1+y| + c_1 = e^x + c_2$,

so $\ln|1+y| = e^x + c$, where $c := c_2 - c_1$

$$\exp(\ln|1+y|) = \exp(e^x + c) = \exp(\exp(x) + c)$$

$$|1+y| = e^{e^x + c} = Ae^{(e^x)}, \quad A := e^c.$$

because $y > -1$. $\longrightarrow$ $1+y = Ae^{(e^x)}$ $\Rightarrow$ $\boxed{y = Ae^{(e^x)} - 1.}$

$\boxed{\text{Example 1, ct'd}}$ Check if $y := A\exp(e^x) - 1$ solves

$$\frac{dy}{dx} = (1+y)e^x :$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[Ae^{(e^x)} - 1\right] = A\frac{d}{dx}\left(e^{(e^x)}\right)$$

$$= Ae^{(e^x)}\frac{d}{dx}\left[e^x\right] \quad \text{Chain rule}$$

$$= Ae^{(e^x)}e^x$$

$$= e^x\left(Ae^{(e^x)}\right)$$

$$= e^x(y+1). \quad \checkmark$$

$\boxed{\text{Example } 2,\ p.\,432}$ Solve $y(x+1)\dfrac{dy}{dx} = x(y^2+1)$.

$$\frac{y}{y^2+1}\,\frac{dy}{dx} = \frac{x}{x+1}$$

$$\int \frac{y}{y^2+1}\,\frac{dy}{dx}\,dx = \int \frac{x}{x+1}\,dx$$

$$\int \frac{y}{y^2+1}\,dy = \int \frac{x}{x+1}\,dx$$

$$\int \frac{y}{y^2+1}\,dy = \int \frac{1}{2u}\,du = \frac{1}{2}\int \frac{1}{u}\,du$$

$u := y^2+1$ $\qquad = \frac{1}{2}\ln|u| + c_1$

$du = 2y\,dy$

$\Rightarrow dy = \frac{1}{2y}\,du$ $\qquad = \frac{1}{2}\ln|y^2+1| + c_1$

$$\int \frac{x}{x+1}\,dx = \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$u = x+1 \Rightarrow x = u-1$ $\qquad = u - \ln|u| + c_2$

$du = dx$ $\qquad = x+1-\ln|x+1| + c_2$

note: $y^2+1>0$ always

$$\frac{1}{2}\ln\{y^2+1\} = x - \ln|x+1| + C, \quad \text{where } C = 1 + c_2 - c_1$$

$$y^2+1 = A\exp(2x - 2\ln|x+1|) = Ae^{2x}e^{-\ln|x+1|}$$

$$= Ae^{2x}\left(e^{\ln|x+1|}\right)^{-1}$$

$\boxed{\text{Example 2, ct'd}}$

$$y^2+1 = A\,\frac{e^{2x}}{e^{\ln|x+1|}}$$

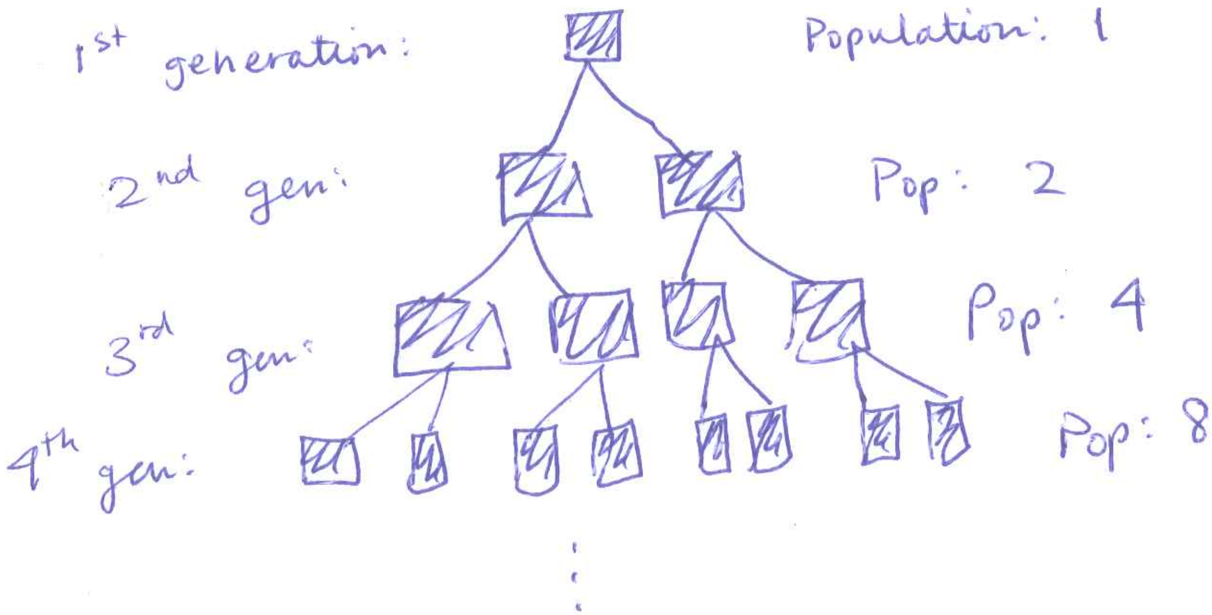
$$y^2+1 = \frac{Ae^{2x}}{|x+1|}$$

$$y^2 = \frac{Ae^{2x}}{|x+1|} - 1$$

$$y = \pm\sqrt{\frac{Ae^{2x}}{|x+1|} - 1}$$

Population growth.

For example, suppose each individual has (on average) two offspring:



Each generation has twice the population of the generation before it.

This is exponential growth!

Usually, we model large populations assuming not discrete (integer) values for population, but continuous ones.

Example
3, p.433

Biomass of yeast culture initially 29 grams. After 30 minutes, the mass is 37 g.

Assuming the eq'n for growth fits the reality when the biomass is below 100g, what is the time it takes for the population (mass) to double from original size?

Sol. The eq'n for exponential growth is

$$y(t) = y_0 e^{kt}, \quad \text{for some } y_0, \text{ some } k.$$

We know $y(0) = 29 \Rightarrow y_0 = 29$, so $y(t) = 29e^{kt}$.

We also know $y(30) = 37$.

$$y(30) = 29e^{k\cdot 30}$$

So $37 = 29e^{30k} \iff \frac{37}{29} = e^{30k}$

$$\iff \ln\left(\tfrac{37}{29}\right) = 30k$$

$$\iff k = \frac{1}{30}\ln\left(\frac{37}{29}\right) = \frac{1}{30}\left(\ln(37) - \ln(29)\right)$$

Then $y(t) = 29e^{\frac{1}{30}\ln\left(\frac{37}{29}\right)t}$ $\qquad e^{a\cdot b} = (e^a)^b$

$$= 29\left(e^{\ln\left(\frac{37}{29}\right)}\right)^{t/30}$$

So $y(t) = 29\left(\frac{37}{29}\right)^{t/30} = \dfrac{37^{t/30}}{29^{t/30-1}}$

Example 3

ct'd

$$y(t) = 29\left(\frac{37}{29}\right)^{t/30}.$$

Q: what t-value gives

$y(t) = 2 \cdot 29$?

$$2 \cdot \cancel{29} = \cancel{29}\left(\frac{37}{29}\right)^{t/30}$$

$$2 = \left(\frac{37}{29}\right)^{t/30}$$

$$\ln(2) = \ln\left(\left(\frac{37}{29}\right)^{t/30}\right)$$

$\ln(a^b) = b\ln(a)$

$$\ln(2) = \frac{t}{30}\ln\left(\frac{37}{29}\right)$$

$$t = 30\,\frac{\ln(2)}{\ln(37/29)} \approx \boxed{85.38 \text{ minutes}}$$

Example 4, p. 434

Diseases die out in a population when treated well.

Suppose that in the course of a given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years to reduce to 1000?

After 1 year, # of cases = 80% of 10,000 = 8,000

" 2 years, — " — = 0.8 (8,000) = 6,400

" 3 — " — = 0.8 (6,400) =

⋮

Model: $y(t) = y_0 e^{-kt}$, $k > 0$, $y_0 > 0$.

For us, $y_0 = 10,000$.

$$\left.\begin{aligned} y(1) &= 0.8(10,000) = 8,000 \\ y(1) &= 10,000 e^{-k \cdot 1} \end{aligned}\right\} \text{implies} \quad \begin{aligned} 8,000 &= 10,000 e^{-k} \\ 0.8 &= e^{-k} \\ \ln(0.8) &= -k \end{aligned}$$

This means $y(t) = 10,000 e^{\ln(0.8) t}$

$= 10,000 (0.8)^t$.

Aside.
$\ln(a) = b$ if & only if $e^b = a$

Set $1,000 = 10,000 (0.8)^t$.

$0.1 = (0.8)^t$

$\ln(0.1) = t \ln(0.8)$

$t = \dfrac{\ln(0.1)}{\ln(0.8)} \approx 10.32$ years

After 1 year, $y(1) = 0.8\,(10{,}000)$

" 2 years, $y(2) = 0.8\,(0.8\,(10{,}000))$

$= (0.8)^2\, 10{,}000$

" 3 years, $y(3) = (0.8)^3\, 10{,}000$

$\vdots$

Radioactive decay.

DEF. The half-life of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay.

The half-life is given by:

$$\text{half-life} = \frac{\ln(2)}{k}.$$

For example, ~~with~~ radon-222 gas follows the exponential decay model:

$$y(t) = y_0 e^{-0.18t}$$

and so the half-life of radon-222 is:

$$\text{half-life} = \frac{\ln(2)}{0.18} \approx 3.9 \text{ days}.$$

Example 5, p.435

Carbon-dating: carbon-14 has a half-life of 5730 years. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Find a formula for half-life of a substance that decays exponentially.

$$y(t) = y_0 e^{kt} \quad , \quad k < 0.$$

So set: $\frac{\cancel{y_0}}{2} = \cancel{y_0} e^{kt}$

$$\ln\left(\tfrac{1}{2}\right) = \ln\left(e^{kt}\right)$$

$\ln(a^b) = b \ln(a)$

$$-\ln(2) = kt$$

$$t = \boxed{-\frac{\ln(2)}{k}}$$

$\boxed{\text{Example 5, ct'd}}$ $\frac{1}{2}$-life = 5730 years. Age of sample in which 10% has decayed?

$\frac{1}{2}$-life = 5730 $\Rightarrow$ $5730 = \dfrac{\ln(2)}{k}$

$\Rightarrow$ $k = \dfrac{\ln(2)}{5730}$

$k \approx 1.2 \times 10^{-4}$

0.00012.

then the model for decay:

$$y(t) = y_0 e^{\left(\frac{\ln 2}{5730}\right)t}$$

Solve: $0.9 \cancel{y_0} = \cancel{y_0} e^{\frac{\ln 2}{5730} \cdot t}$

$$\ln(0.9) = \frac{\ln(2)}{5730} t$$

$$t = \frac{\ln(0.9) \cdot 5730}{\ln(2)}$$

$\approx$ 871 years.

Heat transfer: Newton's Law of Cooling.

LAW NLC. The rate at which an object's temperature changes is roughly proportional to the difference between its temperature and the ambient temperature.

$H(t) :=$ temperature, time t

$H_s :=$ temp. of surroundings (constant)

Then the governing eq'n:

$$\boxed{\frac{dH}{dt} = -k(H - H_s).}$$

Let $y := H - H_s$. Then $\frac{dy}{dt} = \frac{d}{dt}[H - H_s]$

$y(t) := H(t) - H_s$

$$= \frac{dH}{dt} - \cancel{\frac{d}{dt}(H_s)} \to 0$$

$$= -k(H - H_s)$$

$$= -ky.$$

$\frac{dy}{dt} = -ky \Rightarrow y = y_0 e^{-kt}$

So heating/cooling follows: initial temp. $H(0) =: H_0$

$$\boxed{H - H_s = (H_0 - H_s)e^{-kt}.}$$

$\boxed{\text{Example. } 6, \text{ p.} 436}$ A hard-boiled egg at 98°C is put into 18°C water. After 5 min, egg is at 38°C. How long until egg is at 20°C?

Model: $H - H_s = (H_0 - H_s)e^{-kt}$

For us, $H_s = 18\,°C$, $H_0 = 98°C$

so $H - 18 = 80\,e^{-kt}$

① Solve for k

$H(5) = 38°C$

$H(5) - 18 = 80e^{-k \cdot 5}$

$\Rightarrow \quad 38 - 18 = 80\,e^{-5k}$

$20 = 80e^{-5k}$

$\frac{1}{4} = e^{-5k}$

$-\ln 4 = -5k$

$k = \frac{\ln(4)}{5}$

$\boxed{\text{Ex. 6 ct'd}}$

$$H - 18 = 80e^{-\frac{\ln 4}{5}t}$$

$$H = 80e^{-\frac{(\ln 4)t}{5}} + 18.$$

$$20 = 80e^{-\frac{\ln 4}{5}t} + 18$$

$$2 = 80e^{-\frac{\ln 4}{5}t}$$

$$\frac{1}{40} = e^{-\frac{\ln 4}{5}t}$$

$$-\ln(40) = -\frac{\ln(4)}{5}t$$

$$t = \frac{5\ln(40)}{\ln(4)} \approx \boxed{13 \text{ minutes.}}$$