Calculus II
E1 Term, Sections E101 and E196
Instructor: E.M. Kiley
Due June 3, 2016

Week 2: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it. An example of acceptable homework solutions is posted on myWPI under "Course Materials".

Reading

Please read Section 5.5 in time for Tuesday's lecture, and Section 5.6 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 5, "Questions to Guide Your Review", p. 529, Problems 11–15

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 5.5, Problems 1–41 odd
- Section 5.6, Problems 1–39 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Week 2: Homework Problems

Due date: Friday, June 3, 2015, 11:59 p.m. EDT. Please upload a `.pdf` version to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly using correct English.

III) Show your work. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. [10 pts] Show that the function

$$y = \int_0^x 1 + 2\sqrt{\sec t}\ \mathrm{d}t$$

solves the initial value problem

$$\begin{cases} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \tan x \sqrt{\sec x}, \\ y'(0) = 3, \\ y(0) = 0. \end{cases}$$

[Hint: Do *not* solve the initial value problem! Just verify that the given y function is a solution by differentiating a sufficient number of time, and by checking both of the initial conditions.]

Solution. First, we check to see if the differential equation is satisfied. By the Fundamental Theorem of Calculus, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x 1 + 2\sqrt{\sec t}\ \mathrm{d}t\right] = 1 + 2\sqrt{\sec x},$$

and so

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1 + 2\sqrt{\sec x}\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1 + \frac{2}{\sqrt{\cos x}}\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1 + 2(\cos x)^{-1/2}\right] \\ &= 2\left(-\frac{1}{2}\right)(\cos x)^{-3/2}(-\sin x) \\ &= \frac{\sin x}{\cos x\sqrt{\cos x}} \\ &= \tan x\sqrt{\sec x}. \end{aligned}$$

To verify the initial conditions, observe that $y(0) = \int_0^0 1 + 2\sqrt{\sec t}\ \mathrm{d}t = 0$ by the zero-width interval rule. Also, $y'(0) = 1 + 2\sqrt{\sec 0} = 1 + 2(1) = 3$, and so both initial conditions are satisfied.

Problem 2. Evaluate the following definite integrals by first finding the antiderivative using indefinite integration and substitution. Because you will perform substitution on an **in**definite integral, *there will be no bounds to transform.* **You must show every step of your work clearly.**

(a) **[5 points]** $\displaystyle\int_2^4 \frac{1}{x(\ln x)^2}\ \mathrm{d}x$

Solution. Use the substitution $u := \ln x$, so that $\mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}x}[\ln x]\ \mathrm{d}x = \frac{\mathrm{d}x}{x}$. We obtain

$$\int \frac{1}{x(\ln x)^2}\ \mathrm{d}x = \int \frac{\mathrm{d}u}{u^2} = \int u^{-2}\ \mathrm{d}u = \frac{1}{-1}u^{-1} = -\frac{1}{u} = -\frac{1}{\ln x}.$$

For the definite integral, this implies

$$\int_2^4 \frac{1}{x(\ln x)^2}\ \mathrm{d}x = \left[-\frac{1}{\ln x}\right]_2^4 = -\frac{1}{\ln 4} - \left(-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2} - \frac{1}{\ln 4}.$$

(b) **[5 points]** $\displaystyle\int_0^1 x\sqrt{4+5x}\ \mathrm{d}x$

Solution. Using the substitution $u := 4 + 5x$, we obtain $x = \frac{1}{5}u - \frac{4}{5}$, and $\mathrm{d}x = \frac{1}{5}\ \mathrm{d}u$. The integral becomes

$$\begin{aligned}\int x\sqrt{4+5x}\ \mathrm{d}x &= \int \left(\frac{u}{5} - \frac{4}{5}\right)\sqrt{u}\cdot\frac{1}{5}\ \mathrm{d}u = \frac{1}{25}\int (u-4)u^{1/2}\ \mathrm{d}u = \frac{1}{25}\int u^{3/2} - 4u^{1/2}\ \mathrm{d}u \\ &= \frac{1}{25}\left[\frac{2}{5}u^{5/2} - 4\frac{2}{3}u^{3/2}\right] = \frac{2}{125}u^{5/2} - \frac{8}{75}u^{3/2} = \frac{2}{125}(4+5x)^{5/2} - \frac{8}{75}(4+5x)^{3/2}.\end{aligned}$$

For the definite integral, this implies

$$\begin{aligned}\int_0^1 x\sqrt{4+5x}\ \mathrm{d}x &= \left[\frac{2}{125}(4+5x)^{5/2} - \frac{8}{75}(4+5x)^{3/2}\right]_0^1 = \left[\frac{2\cdot 9^{5/2}}{125} - \frac{8\cdot 9^{3/2}}{75}\right] - \left[\frac{2\cdot 4^{5/2}}{125} - \frac{8\cdot 4^{3/2}}{75}\right] \\ &= \frac{2\cdot 3^5}{125} - \frac{8\cdot 3^3}{75} - \frac{2\cdot 2^5}{125} + \frac{8\cdot 2^3}{75} = \frac{2(3^5-2^5)}{125} - \frac{8(3^3-2^3)}{75} = \frac{6(3^5-2^5) - 40(3^3-2^3)}{375} \\ &= \frac{506}{375} \approx 1.3493.\end{aligned}$$

(c) **[5 points]** $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\ \mathrm{d}\theta$

Solution. Use the substitution $u = \theta^{3/2}$, so that $\mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}\theta}[\theta^{3/2}]\ \mathrm{d}\theta\frac{3}{2}\theta^{1/2}\ \mathrm{d}\theta = \frac{3}{2}\sqrt{\theta}\ \mathrm{d}\theta$. Then the integral becomes

$$\begin{aligned}\int \sqrt{\theta}\cos^2(\theta^{3/2})\ \mathrm{d}\theta &= \frac{2}{3}\int \cos^2 u\ \mathrm{d}u = \frac{2}{3}\int \frac{1+\cos(2u)}{2}\ \mathrm{d}u = \frac{1}{3}\int 1 + \cos(2u)\ \mathrm{d}u \\ &= \frac{1}{3}\left[u + \frac{1}{2}\sin(2u)\right] = \frac{1}{3}\theta^{3/2} + \frac{1}{6}\sin(2\theta^{3/2}).\end{aligned}$$

For the definite integral, this implies

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\ \mathrm{d}\theta = \left[\frac{1}{3}\theta^{3/2} + \frac{1}{6}\sin(2\theta^{3/2})\right]_0^{\sqrt[3]{\pi^2}} = \left[\frac{\pi}{3} + \frac{1}{6}\sin(2\pi)\right] - [0 + \sin(0)] = \frac{\pi}{3}.$$

Problem 3. Evaluate the following definite integrals using substitution. *You will need to transform the bounds.* **You must show every step of your work clearly.**

(a) **[5 points]** $\displaystyle\int_0^\pi 3\cos^2 x\sin x\ \mathrm{d}x$

Solution. Use the substitution $u = \cos x$, so $\mathrm{d}u = -\sin x \, \mathrm{d}x$, $u(0) = 1$, $u(\pi) = -1$. Then

$$\int_0^\pi 3\cos^2 x \sin x \ \mathrm{d}x = \int_1^{-1} -3u^2 \ \mathrm{d}u = \int_{-1}^{1} 3u^2 \ \mathrm{d}u = u^3\big|_{-1}^{1} = 1^3 - (-1)^3 = 1 - (-1) = 1 + 1 = 2.$$

(b) [5 points] $\displaystyle\int_{\pi/4}^{\pi/2} \cot t \ \mathrm{d}t$

Solution. Well, $\cot t = \frac{\cos t}{\sin t}$, so we can use the substitution $u := \sin t$. Then $\mathrm{d}u = \cos t \ \mathrm{d}t$, $u(\pi/2) = \sin(\pi/2) = 1$, and $u(\pi/4) = \sin(/pi/4) = \frac{1}{\sqrt{2}}$. Then the integral becomes

$$\int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t} \ \mathrm{d}t = \int_{1/\sqrt{2}}^{1} \frac{\mathrm{d}u}{u} = [\ln|u|]_{1/\sqrt{2}}^{1} = \ln(1) - \ln(1/\sqrt{2}) = -\ln(2^{-1/2}) = -1 \cdot \left(-\frac{1}{2}\right) \ln 2 = \frac{\ln(2)}{2}.$$

(c) [5 points] $\displaystyle\int_0^1 \frac{10\sqrt{x}}{(1+x^{3/2})^2} \ \mathrm{d}x$

Solution. Use the substitution $u := 1 + x^{3/2}$, so $\mathrm{d}u = \frac{3}{2}\sqrt{x} \ \mathrm{d}x$, $u(0) = 1$ and $u(1) = 2$. Then the integral becomes

$$\int_0^1 \frac{10\sqrt{x}}{(1+x^{3/2})^2} \ \mathrm{d}x = \int_1^2 \frac{10 \cdot \frac{2}{3} \ \mathrm{d}u}{u^2} = \frac{20}{3} \int_1^2 u^{-2} \ \mathrm{d}u = \frac{20}{3}\left[-\frac{1}{u}\right]_1^2 = \frac{20}{3}\left[-\frac{1}{2} + \frac{1}{1}\right] = \frac{20}{3} \cdot \frac{1}{2} = \frac{10}{3}.$$

Problem 4. Use Theorem 8 about the integrals of even and odd functions over symmetric intervals to compute the following definite integrals. **You must prove that the integrand is even or odd, and you must state that the interval is symmetric.**

(a) [5 points] $\displaystyle\int_{-2}^{2} x\sqrt{4-x^2} \ \mathrm{d}x$

Solution. Note that the interval is symmetric about 0, and that if $f(x) := x\sqrt{4-x^2}$, then $f(-x) = -x\sqrt{4-(-x)^2} = -x\sqrt{4-x^2} = -f(x)$, which implies that the integrand is odd. Therefore, we have

$$\int_{-2}^{2} x\sqrt{4-x^2} \ \mathrm{d}x = 0.$$

(b) [5 points] $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta}{1+(\sin\theta)^2} \ \mathrm{d}\theta$

Solution. Note that the interval is symmetric about 0, and that if $f(\theta) := \frac{2\cos\theta}{1+(\sin\theta)^2}$, then $f(-\theta) = \frac{2\cos(-\theta)}{1+(\sin(-\theta))^2} = \frac{2\cos\theta}{1+(\sin\theta)^2} = f(\theta)$, and so the integrand is even. Therefore, we have

$$\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta}{1+(\sin\theta)^2} \ \mathrm{d}\theta = 2\int_0^{\pi/2} \frac{2\cos\theta}{1+\sin^2\theta} \ \mathrm{d}\theta.$$

We use the substitution $u := \sin\theta$, so that $\mathrm{d}u = \cos\theta \ \mathrm{d}\theta$, $u(0) = 0$, and $u(\pi/2) = 1$. The integral becomes

$$2\int_0^{\pi/2} \frac{2\cos\theta}{1+\sin^2\theta} \ \mathrm{d}\theta = 2\int_0^1 \frac{\mathrm{d}u}{1+u^2} = 2\left[\arctan(u)\right]_0^1 = 2\left[\arctan(1) - \arctan(0)\right] = 2\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2}.$$

Problem 5. [10 points] Find the area of the region enclosed by the curves $y = 2\sin x$ and $y = \sin(2x)$, for $x \in [0, \pi]$.

Solution. We first observe that for $x \in [0, \pi]$, $2\sin x \geq \sin(2x)$. This implies that the area is the integral of the difference:

$$\begin{aligned}
A &= \int_0^\pi 2\sin x - \sin(2x)\ \mathrm{d}x \\
&= \left[-2\cos x + \frac{1}{2}\cos(2x)\right]_0^\pi \\
&= \left[-2\cos\pi + \frac{1}{2}\cos(2\pi)\right] - \left[-2\cos(0) + \frac{1}{2}\cos(0)\right] \\
&= -2(-1) + \frac{1}{2} + 2 - \frac{1}{2} = 4.
\end{aligned}$$