Calculus II
E1 Term, Sections E101 and E196
Instructor: E.M. Kiley
Due June 6, 2016

Week 3: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it. An example of acceptable homework solutions is posted on myWPI under "Course Materials".

Reading

Please read Section 5.5 in time for Tuesday's lecture, and Section 5.6 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 6, "Questions to Guide Your Review", p. 415, Problems 1, 2, 4, 5

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 6.1, Problems 1–49 odd
- Section 6.3, Problems 1–25 odd; 26
- Section 6.4, Problems 5–17 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Week 6: Homework Problems

Due date: Monday, June 6, 2015, 11:59 p.m. EDT. Please upload a `.pdf` version to myWPI (`my.wpi.edu`).

> **Rules for Calculus Assignments**:
>
> **I)** Each student must compose his or her assignments independently. However, brainstorming may be done in groups.
>
> **II)** Please typeset your solutions using LaTeX, or handwrite them neatly and legibly using correct English.
>
> **III)** Show your work. Explain your answers using **full English sentences**.
>
> **IV) No late assignments will be accepted for credit.**

Problem 1. Currently, the average annual rate of interest on Vanguard's VBLTX (long-term bond index fund) is 7.67%.[2]

(a) Suppose that you invest \$5,500 at one time this year in VBLTX, and that you stop contributing to this account afterward. If interest is compounded annually, and if the annual rate of interest stays fixed at 7.67% per year, how much money will you have in that account forty years from now? (See Example 8 on Page 39.)

> **Solution.** You will have $\$5,500 \cdot 1.0767^{40} \approx \$105,718.55$ in the account at the end of forty years.

(b) Suppose that you wait ten years before investing your \$5,500. How much money will you have forty years from now in that case? (That is, how much will you have thirty years after you start investing?)

> **Solution.** If you wait 10 years before making your deposit, you will have $\$5,500 \cdot 1.0767^{40} \approx \$50,489.85$ in the account forty years from now—less than half of what you would have had if you'd made that initial deposit now.

(c) Suppose that beginning now, you invest \$5,500 per year in VBLTX for the next fifteen years (you deposit a total of fifteen times). How much money will you deposit over that period of time? (Do not compute interest yet.) If the interest rate remains fixed at 7.67% and interest is compounded annually, compute the interest to find out how much money will be in your account at the end of the fifteenth year.

Hint: The amount of money in your account will be the sum of fifteen terms:

$$\sum_{i=1}^{15} 5500 \cdot 1.0767^i = \underbrace{5500 \cdot 1.0767^1}_{\text{last deposit's contribution}} + 5500 \cdot 1.0767^2 + \cdots + 5500 \cdot 1.0767^{14} + \underbrace{5500 \cdot 1.0767^{15}}_{\text{first deposit's contribution}}.$$

You might find a tool like WolframAlpha useful in computing this sum; use this example as a template: `http://www.wolframalpha.com/input/?i=%5Csum_%7Bi%3D0%7D%5E%7B20%7D+43*1.06%5Ei`, and change the numbers in it to obtain the sum above (the sum is under the "Decimal Form" heading).

> **Solution.** Over the next fifteen years, you will deposit $\$5,500 \cdot 15 = \$82,500$ into the account, and will end up with $\sum_{i=1}^{15} \$5,500 \cdot 1.0767^i \approx \$156,720.33$ in the account at the end of those fifteen years.

[2] The average is taken over all the years that Vanguard has had this fund available for customers to invest in—that is, since March 1994. You can see the return rates for some of Vanguard's other mutual funds here: `https://investor.vanguard.com/mutual-funds/vanguard-mutual-funds-list`. The returns on some of their funds are lower than those on VBLTX, and others are higher.

(d) Continuing the scenario in part (c), suppose that you never make another contribution to that account after those first fifteen years go by, but that the interest rate remains fixed at the same rate of 7.67% and compounds annually for the next thirty years after. How much will be in that account after those next thirty years have passed?

Solution. Thirty years after you make your final deposit, the amount in the account will be $\$156,720.33 \cdot 1.0767^{30} \approx \$1,438,690.00$. Note that the interest generated on this principal alone will be $0.0767 \cdot \$1,438,690.00 \approx \$110,347.52$ per year, which I think is a pretty decent retirement salary.

(e) Suppose that your friend never thinks very much about his retirement now, and after his mid-life crisis in fifteen years, he begins to invest $5,500 per year in VBLTX, with the same fixed 7.67% rate of annually compounded interest, for the next thirty years after. How much money will he deposit in total? How much money will be in his account forty-five years from now, when you are both ready to retire? (Similar Hint: The total amount in his account at that time will be the sum of thirty terms—do this with WolframAlpha like you did part (c).)[3]

Solution. Your friend will deposit $\$5,500 \cdot 30 = \$165,000$ in total—exactly twice as much as you do—and will end up with $\sum_{i=1}^{30} \$5,500 \cdot 1.0767^i \approx \$631,558.95$, less than half (43%) as much as you will.

Problem 2. **[20 points]** Use a definite integral to verify the familiar formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a right circular cone with base radius r and height h.

Solution. One way to do this is to represent the cone as the rotation of the line

$$y = \frac{r}{h}x$$

about the x-axis. (This means the cone's vertex is at the origin.) The radius is therefore

$$R(x) = \frac{rx}{h},$$

for x-values in the interval $[0, h]$. The volume of the cone is therefore

$$V = \int_0^h \pi \left(\frac{rx}{h}\right)^2 \, dx = \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h^3}{3h^2} - 0 = \frac{1}{3}\pi r^2 h,$$

just as desired.
You might have used a different line (for example, you might have assumed the base of the cone was centered on the origin, and the vertex was at $(h, 0)$), but the definite integral—when you choose the bounds correctly—works out to be the same (it has to, because it represents the same volume).

Problem 3. **[20 points]** A manufacturer needs to make corrugated metal sheets 36 inches wide with cross-sections in the shape of the curve $y = \frac{1}{2}\sin(\pi x)$, for $0 \leq x \leq 36$. How wide must the original flat sheets be, in order for the manufacturer to produce these corrugated sheets?

[3] I hope that doing this problem makes you think about how important it is to begin investing early. The student loan interest you pay won't be compounding for the rest of your life, but the interest your earn on your savings will—and, to boot, yearly IRA contributions are currently limited by U.S. law (to $5,500 per year for people in your instructor's age and income bracket), making it truly impossible to catch up in the future if you fall behind now. So, learn more about IRAs here: `http://money.cnn.com/retirement/guide/IRA_Basics.moneymag/index.htm`, and begin investing while you're young (:

Solution. This question was asking you to find the arc length of the curve. Start by computing the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\sin(\pi x)\right] = \frac{\pi}{2}\cos(\pi x),$$

so that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{\pi^2}{4}\cos^2(\pi x),$$

and

$$1 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 - \frac{\pi^2}{4}\cos^2(\pi x) = \frac{4 - \pi^2\cos^2(\pi x)}{4}.$$

Therefore, the arc length is

$$L = \int_0^{36} \sqrt{\frac{4 - \pi^2\cos^2(\pi x)}{4}}\ \mathrm{d}x = \frac{1}{2}\int_0^{36} \sqrt{4 - \pi^2\cos^2(\pi x)}\ \mathrm{d}x.$$

This integral is not evaluable by standard techniques, and must be approximated. I didn't expect you to do this, but the approximate value of the integral, using a left-hand Riemann sum, is 52.6. Therefore, the manufacturer needs 52.6 inches of material to make his 36-inch corrugated metal sheet.

Problem 4. The astroid shown in the figure below has equation $x^{2/3} + y^{2/3} = 1$.

(a) **[10 points]** Find the total length of the astroid.

Solution. We solve for x in terms of y for the astroid (because we will need this in part (b)). That is,

$$x^{2/3} + y^{2/3} = 1 \iff x^{2/3} = 1 - y^{2/3} \iff x = \left(1 - y^{2/3}\right)^{3/2}.$$

Therefore,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(1 - y^{2/3}\right)^{3/2}\right] = \frac{3}{2}\left(1 - y^{2/3}\right)^{1/2} \cdot \left(-\frac{2}{3}y^{-1/3}\right) = \frac{-\sqrt{1 - y^{2/3}}}{\sqrt[3]{y}}.$$

Then

$$\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = \left(\frac{-\sqrt{1 - y^{2/3}}}{\sqrt[3]{y}}\right)^2 = \frac{1 - y^{2/3}}{y^{2/3}} = \frac{1}{y^{2/3}} - 1 = y^{-2/3} - 1,$$

and

$$\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} = \sqrt{1 + y^{-2/3} - 1} = \sqrt{y^{-2/3}} = y^{-1/3}.$$

Due to symmetry, the total length of the astroid is four times the length in one quadrant, which is described when the y-values vary from 0 to 1. Therefore, the total length of the astroid is given by the formula

$$L = 4\int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y = 4\int_0^1 y^{-1/3}\ \mathrm{d}y = 4 \cdot \frac{3}{2}\left[y^{2/3}\right]_0^1 = 6\,[1 - 0] = 6.$$

(b) **[10 points]** Find the area of the surface of revolution generated by rotating the astroid around the y-axis.

Solution. We already computed $\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} = y^{-1/3}$ in the previous problem. Again due to symmetry, the total surface area of the astroid will be twice that in the half represented by rotating the portion in the first quadrant about the axis—so the formula for the surface area when revolving about the y-axis is

$$A = 2 \cdot 2\pi \int_0^1 x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}x = 4\pi \int_0^1 \left(1-y^{2/3}\right)^{3/2} y^{-1/3}\ \mathrm{d}y.$$

This integral can be computed using the substitution $u := y^{2/3},\ \mathrm{d}u = \frac{2}{3}y^{-1/3}\ \mathrm{d}y$, $u(0)=0$, $u(1)=1$. With this substitution, the integral transforms into

$$A = 4\pi \int_0^1 (1-u)^{3/2} \cdot \frac{3}{2}\ \mathrm{d}u = 6\pi \int_0^1 (1-u)^{3/2}\ \mathrm{d}u.$$

We apply the second substitution $v := 1-u,\ \mathrm{d}v = -\ \mathrm{d}u$, $v(0)=1$, $v(1)=0$, to obtain

$$A = -6\pi \int_1^0 v^{3/2}\ \mathrm{d}v = 6\pi \int_0^1 v^{3/2}\ \mathrm{d}v = 6\pi \cdot \frac{2}{5}\left[v^{5/2}\right]_0^1 = \frac{12\pi}{5}[1-0] = \frac{12\pi}{5}.$$