Section 4.5 : Indeterminate forms and l'Hôpital's Rule.

- Indeterminate form "0/0"
- L'Hôpital's rule and its use
- Indeterminate forms "∞/∞", "$\infty \cdot 0$", "$\infty - \infty$"
- Indeterminate powers "1^∞", "0^0", "∞^0"

- Indeterminate form "0/0"

$$\boxed{\text{EXAMPLE: How does } F(x) := \frac{x - \sin(x)}{x^3} \text{ behave near } x = 0\text{ ?}}$$

- $F(0)$ is undefined (why?), so we look at $\lim_{x\to 0} F(x)$ instead.
- Temptation is to apply the quotient rule for limits:

$$\lim_{x\to c} \frac{f(x)}{g(x)} \overset{\checkmark}{=} \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)},$$ provided limits on RHS (right-hand side) exist, and denominator of RHS not zero.

So, $$\lim_{x\to 0} \frac{x - \sin(x)}{x^3} \overset{?}{=} \frac{\lim_{x\to 0} x - \sin(x)}{\lim_{x\to 0} x^3}$$

$$\overset{?}{=} \text{"}\frac{0}{0}\text{"}$$

- Such a limit ("$\frac{0}{0}$" form) may or may not exist in general — but we need more tools to examine further!

NOTE: We use "$\frac{0}{0}$" as notation for the INDETERMINATE FORM produced by substituting $x = a$ into the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both continuous functions that are zero at $x = a$.

Lecture 1, continued.

- In general, an INDETERMINATE FORM is a meaningless expression such as "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$\infty \cdot 0$", "$\infty - \infty$", "0^0", "1^∞", which cannot be evaluated in a consistent way.

- Sometimes these limits work out:

EXAMPLE: $f'(a) \stackrel{?}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \stackrel{?}{=} \frac{f(a) - f(a)}{a - a} = \boxed{\text{"}\frac{0}{0}\text{"}}$

- But not always! As we said, we need stronger tools to analyze further...

THEOREM: L'Hôpital's Rule (Johann Bernoulli, 1696)

IF
- $f(x)$ and $g(x)$ are differentiable in a nbd. (neighborhood) of a
- $g'(x) \neq 0$ in a deleted nbd. of a
- EITHER $\left(f(a) = g(a) = 0\right)$ "$\frac{0}{0}$" or $\left(f(a) = g(a) = \infty\right)$ "$\frac{\infty}{\infty}$"

THEN $\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}} \stackrel{?}{=} \underbrace{\lim_{x \to a} \frac{f'(x)}{g'(x)}}_{\text{provided this limit exists.}}$

NOTES:
- Must have "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" form!!

  (i.e., l'Hôpital's Rule doesn't replace the quotient rule for limits)

- We really do want $\frac{f'(x)}{g'(x)}$, NOT $\cancel{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}$

  (i.e., do not incorrectly use the quotient rule for derivatives)

EXAMPLES. $\boxed{1}$ $\lim_{x\to 0} \dfrac{3x - \sin(x)}{x} \cancel{\underset{\text{Quot}}{=}} \dfrac{\lim_{x\to 0} 3x - \sin(x)}{\lim_{x\to 0} x} \cancel{=}$ " $\dfrac{0}{0}$ "

Can we use l'Hôpital's rule?

$$\lim_{x\to \boxed{0}_a} \frac{\boxed{3x - \sin(x)}\ f(x)}{\boxed{x}\ g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}, \quad \text{so} \quad \begin{array}{l} a = 0 \\ f(x) = 3x - \sin(x) \\ g(x) = x \end{array}$$

✓ $f(a) = g(a) = 0$

✓ $\left.\begin{array}{l} f'(x) = 3 - \cos(x) \\ g'(x) = 1 \end{array}\right\}$ both f and g are diff'ble everywhere, including in a nbd. of $a = 0$

✓ $g'(x) = 1 \neq 0$ everywhere, including in a deleted nbd. of $a = 0$.

so, YES – let's apply l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{3x - \sin(x)}{x} \underset{\text{l'Hôp}}{\overset{✓}{=}} \lim_{x\to 0} \frac{3 - \cos(x)}{1}$$

$$= \lim_{x\to 0} 3 - \cos(x)$$

$$= \lim_{x\to 0} 3 - \lim_{x\to 0} \cos(x)$$

$$= 3 - 1$$

$$\boxed{= 2}$$

$\boxed{2}$ $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1}{x} \cancel{\underset{\text{Quot}}{=}} \dfrac{\lim_{x\to 0} \sqrt{1+x} - 1}{\lim_{x\to 0} x} \cancel{=}$ " $\dfrac{0}{0}$ "

$f(x) = \sqrt{1+x} - 1$, $g(x) = x$

$f'(x) = \dfrac{1}{2\sqrt{1+x}}$, $g'(x) = 1$

$$\underset{\text{l'Hôp}}{\overset{?}{=}} \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1}$$

-0.5 0.5

0

✓ $f(0) = g(0) = 0$

✓ f, g diff'ble around $a = 0$

✓ $g'(x) = 1 \neq 0$ around $a = 0$

$$= \lim_{x\to 0} \frac{1}{2\sqrt{1+x}}$$

$$= \frac{1}{2\sqrt{1+0}} = \boxed{\frac{1}{2}}$$

EXAMPLES [3] $\lim_{x\to 0} \frac{x-\sin(x)}{x^3} \overset{?}{\underset{Q.R.}{\cancel{=}}} \frac{\lim_{x\to 0} x-\sin(x)}{\lim_{x\to 0} x^3}$

$\overset{?}{=}$ " $\frac{0}{0}$ "

use l'Hôpital's Rule ?

check : $f(x) = x - \sin(x)$, $g(x) = x^3$

- $f'(x) = 1-\cos(x)$, $g'(x) = 3x^2$ } so both are diff'ble around $a=0$
- $g'(x) = 3x^2 \neq 0$ in a deleted nbd. of $a=0$
- $f(0) = g(0) = 0$

USE L'HÔPITAL'S RULE.

$\lim_{x\to 0} \frac{x-\sin(x)}{x^3} \overset{\text{L'Hôp.}\ ?}{=} \boxed{\lim_{x\to 0} \frac{1-\cos(x)}{3x^2}} \overset{\text{Quot}\ ?}{\cancel{=}} \frac{\lim_{x\to 0} 1-\cos(x)}{\lim_{x\to 0} 3x^2} \cancel{=}$ " $\frac{0}{0}$ "

use l'Hôpital again?

$f(x) = 1-\cos(x) \Rightarrow f'(x) = \sin(x)$

$g(x) = 3x^2 \Rightarrow g'(x) = 6x$

- ✓ f, g both diff'ble around $a=0$
- ✓ $g'(x) = 6x \neq 0$ around $a=0$ except at $a=0$
- ✓ $g(0) = f(0) = 0$

$\overset{?}{\underset{\text{L'Hôp}}{=}} \lim_{x\to 0} \frac{\sin(x)}{6x} \overset{?}{\underset{\text{Quot}}{\cancel{=}}} \frac{\lim_{x\to 0} \sin(x)}{\lim_{x\to 0} 6x} \cancel{=}$ " $\frac{0}{0}$ "

Use l'Hôpital a 3rd time ?

$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$

$g(x) = 6x \Rightarrow g'(x) = 6$

- ✓ f, g both diff'ble around $a=0$
- ✓ $g'(x) = 6 \neq 0$ around $a=0$
- ✓ $g(0) = f(0) = 0$

$\overset{?}{\underset{\text{L'Hôp}}{=}} \lim_{x\to 0} \frac{\cos(x)}{6} = \frac{1}{6} \lim_{x\to 0} \cos(x) = \boxed{\frac{1}{6}}$.

EXAMPLES. $\boxed{4}$ $\lim_{x\to 0} \frac{1-\cos(x)}{x+x^2} \underset{\text{Quot}}{\cancel{=}} \frac{\lim_{x\to 0} 1-\cos(x)}{\lim_{x\to 0} x+x^2} = \text{"}\frac{0}{0}\text{"}$

$\underset{\text{l'Hôp}}{\overset{\checkmark}{=}} \lim_{x\to 0} \frac{\sin(x)}{1+2x} \underset{\text{Quot}}{\overset{\checkmark}{=}} \frac{\lim_{x\to 0} \sin(x)}{\lim_{x\to 0} 1+2x} = \frac{0}{1} \boxed{=0}$

A FALSE EXAMPLE. $\boxed{5}$ $\lim_{x\to 0} \frac{\sin(x)}{1+2x}$ $\quad f(x)=\sin(x)$, $g(x)=1+2x$
$f'(x)=\cos(x)$, $g'(x)=2$

"$\frac{0}{0}$" "$\frac{\infty}{\infty}$"

- $f(0)=\sin(0)=0$
- $g(0)=1+2(0)=1$

1ST CONDITION NOT SATISFIED (since $g(0)\neq 0$)

✓ • f, g both diff'ble around $a=0$

✓ • $g'(x)=2\neq 0$ around $a=0$

what if we used l'Hôpital's rule anyway?

$$\lim_{x\to 0} \frac{\sin(x)}{1+2x} \underset{\text{l'Hôp.}}{\overset{?}{=}} \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2} \underbrace{\lim_{x\to 0} \cos(x)}_{1}$$

$$\cancel{= \frac{1}{2}}$$

INCORRECT!

So we MUST have the indeterminate form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".

EXAMPLE $\boxed{6}$ $\boxed{\lim_{x\to\infty} \frac{e^x}{x^2}}$ $\underset{\text{Quot}}{\overset{?}{\cancel{=}}} \frac{\lim_{x\to\infty} e^x}{\lim_{x\to\infty} x^2} = \text{"}\frac{\infty}{\infty}\text{"}$ $\quad f(x)=e^x$, $g(x)=x^2$
$f'(x)=e^x$, $g'(x)=2x$

✓ • $f(\infty)=g(\infty)=\infty$

✓ • f, g diff'ble

✓ • $g(\infty)\neq 0$

$\underset{\text{l'Hôp}}{\overset{\checkmark ?}{=}} \lim_{x\to\infty} \frac{e^x}{2x} \underset{\text{Quot}}{\cancel{=}} \frac{\lim_{x\to\infty} e^x}{\lim_{x\to\infty} 2x} = \text{"}\frac{\infty}{\infty}\text{"}$ $\quad f(x)=e^x$, $g(x)=2x$
$f'(x)=e^x$, $g'(x)=2$

$\underset{\text{l'Hôp}}{\overset{\checkmark ?}{=}} \lim_{x\to\infty} \frac{e^x}{2} = \frac{1}{2} \lim_{x\to\infty} e^x$

- $f(\infty)=g(\infty)=\infty$
- f', g' exist
- $g(\infty)\neq 0$

$= \frac{1}{2}(\infty) \boxed{=\infty}$

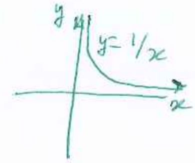
Other indeterminate forms: $0 \cdot \infty$, $\infty - \infty$

We turn them into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic manipulation:

EXAMPLES

7 $\lim_{x\to\infty}\left(x \sin\left(\frac{1}{x}\right)\right) \underset{\text{Prod}}{\cancel{=}} \left(\lim_{x\to\infty} x\right)\left(\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)\right)$

$\underset{\text{Func.}}{\cancel{=}} (\infty)\left(\sin\left(\lim_{x\to\infty}\frac{1}{x}\right)\right)$



$\cancel{=} (\infty)(\sin(0))$

$\cancel{=}$ "$\infty \cdot 0$"

try again: $\lim_{x\to\infty}\left[x \sin\left(\frac{1}{x}\right)\right] = \lim_{h\to 0^+}\left[\left(\frac{1}{h}\right)\sin(h)\right]$ LET $\boxed{h := \frac{1}{x}}$ $x = \frac{1}{h}$

$= \lim_{h\to 0^+} \frac{\sin(h)}{h} \underset{\text{Quot.}}{\overset{?}{=}}$ "$\frac{0}{0}$"

$\underset{\text{l'Hôp}}{\overset{?}{=}} \lim_{h\to 0^+} \frac{\cos(h)}{1}$

$= \lim_{h\to 0^+} \cos(h) = \boxed{1.}$

8 $\lim_{x\to 0^+} \sqrt{x}\,\ln(x) \underset{\text{Prod}}{\cancel{\overset{?}{=}}} \left(\lim_{x\to 0^+}\sqrt{x}\right)\left(\lim_{x\to 0^+}\ln(x)\right) =$ "$0 \cdot \infty$"

$\sqrt{x} = \frac{1}{1/\sqrt{x}}$

$= \lim_{x\to 0^+} \frac{\ln(x)}{1/\sqrt{x}}$

$\underset{\text{Quot}}{\cancel{\overset{?}{=}}} \frac{\lim_{x\to 0^+} \ln(x)}{\lim_{x\to 0^+} 1/\sqrt{x}} =$ "$\frac{\infty}{\infty}$"

So let $f(x) = \ln(x)$, $g(x) = 1/\sqrt{x}$,
$f'(x) = 1/x$, $g'(x) = -1/(2x^{3/2})$

✓ $f(0^+) \to \infty$, $g(0^+) \to \infty$
✓ f, g diff'ble around $a = 0$
✓ $g'(x) = -1/2x^{3/2} \neq 0$ around $a = 0$

$\underset{\text{l'Hôp}}{\overset{?}{=}} \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}}$

$= \lim_{x\to 0^+} \left(\frac{1}{x}\right)\left(-2x^{3/2}\right) = \lim_{x\to 0^+}\left(-2\sqrt{x}\right) = \boxed{0.}$

Examples. $\boxed{9}$ $\lim_{x\to 0}\left[\frac{1}{\sin(x)} - \frac{1}{x}\right]$.

Careful to check both the left and right limits!

$$\lim_{x\to 0^+}\left[\frac{1}{\sin(x)} - \frac{1}{x}\right] \cancel{=} \text{"}\infty - \infty\text{"}$$

$$\lim_{x\to 0^-}\left[\frac{1}{\sin(x)} - \frac{1}{x}\right] \cancel{=} \text{"}-\infty + \infty\text{"}$$

Try again:

$$\lim_{x\to 0}\left[\frac{1}{\sin(x)} - \frac{1}{x}\right] = \lim_{x\to 0} \frac{x - \sin(x)}{x\sin(x)}$$ get a common denominator

"$\frac{0}{0}$"

$$\overset{?}{\underset{\text{l'Hôp}}{=}} \lim_{x\to 0} \frac{1-\cos(x)}{\sin(x) + x\cos(x)}$$ "$\frac{0}{0}$"

$$\overset{?}{\underset{\text{l'Hôp}}{=}} \lim_{x\to 0} \frac{+\sin(x)}{\cos(x) + \cos(x) - x\sin(x)}$$

$$= \lim_{x\to 0} \frac{+\sin(x)}{2\cos(x) - x\sin(x)}$$

$$\overset{?}{\underset{\text{Quot}}{=}} \frac{\lim_{x\to 0} +\sin(x)}{\lim_{x\to 0} 2\cos(x) - x\sin(x)} = \frac{0}{2} = \boxed{0}$$

So far, strategies for "$\infty - \infty$" and "$0\cdot\infty$":

- Get a common denominator
- Write $a\cdot b$ as $\frac{a}{1/b}$, or $\frac{b}{1/a}$
- Rewrite limiting variable as $h := \frac{1}{x}$

Indeterminate powers "1^∞", "0^0", "∞^0".

• Recall : $\boxed{e^{\ln(\text{anything})} = \text{anything}}$; $\boxed{\ln(a^b) = b \cdot \ln(a)}$

example, $\boxed{10}$ $\lim_{x \to 0} (\cos x)^{1/x^2}$ has the form "1^∞".

$$= \exp\left[\ln\left[\lim_{x\to 0} (\cos x)^{1/x^2}\right]\right]$$

$$= \exp\left[\lim_{x\to 0} \ln\left[(\cos x)^{1/x^2}\right]\right]$$ continuity of ln function

$$= \exp\left[\lim_{x\to 0} \frac{1}{x^2} \cdot \ln(\cos x)\right]$$

$$= \exp\left[\lim_{x\to 0} \frac{\ln(\cos x)}{x^2}\right]$$ has form "$\frac{0}{0}$"

$f(x) = \ln(\cos x)$, $g(x) = x^2$

$f'(x) = \frac{-\sin x}{\cos x}$, $g'(x) = 2x$

✓ f, g diff'ble in a nbd. of 0

✓ $g'(x) = 2x \neq 0$ in a deleted nbd of 0.

$$\overset{?}{\underset{\text{l'Hôp}}{=}} \exp\left[\lim_{x\to 0} \frac{-\sin(x)}{2x\cos(x)}\right]$$ has form "$\frac{0}{0}$" again!

$f(x) = -\sin x$, $g(x) = 2x\cos(x)$

$f'(x) = -\cos x$, $g'(x) = 2\cos(x) - 2x\sin(x)$

✓ f, g - diff'ble in nbd. of 0

✓ $g'(x) \neq 0$ in a deleted nbd. of 0.

$$\overset{?}{\underset{\text{l'Hôp}}{=}} \exp\left[\lim_{x\to 0} \frac{-\cos(x)}{2\cos(x) - 2x\sin(x)}\right]$$

$$= \exp\left[-\frac{1}{2}\right]$$

$$= e^{-1/2} = \boxed{\frac{1}{\sqrt{e}} = \lim_{x\to 0} (\cos x)^{1/x^2}}$$

example. [11] $\lim_{x\to\infty} x^{1/x}$ has the form "∞^0"

$$= \exp\left[\ln\left[\lim_{x\to\infty} x^{1/x}\right]\right]$$

$$= \exp\left[\lim_{x\to\infty} \ln(x^{1/x})\right]$$

$$= \exp\left[\lim_{x\to\infty} \frac{1}{x}\ln(x)\right]$$

$$= \exp\left[\lim_{x\to\infty} \frac{\ln(x)}{x}\right]$$ has form "$\frac{\infty}{\infty}$"

let $f(x) = \ln(x)$, $g(x) = x$
$f'(x) = \frac{1}{x}$, $g'(x) = 1$

- f, g diff'ble as $x \to \infty$
- $g'(x) \neq 0$ as $x \to \infty$.

$$\underset{\text{l'Hôp}}{\overset{?}{=}} \exp\left[\lim_{x\to\infty} \frac{1/x}{1}\right]$$

$$= \exp\left[\lim_{x\to\infty} \frac{1}{x}\right]$$

$$= \exp[0]$$

$$= e^0 = \boxed{1 = \lim_{x\to\infty} x^{1/x}}.$$

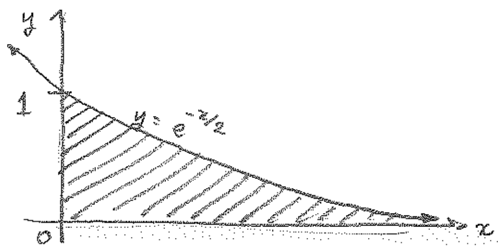
More examples at end of lecture (as time permits)

Section 8.8: Improper Integrals.

- Infinite limits of integration
- The integral $\int_1^\infty \frac{dx}{x^p}$
- Integrands with vertical asymptotes
- Direct comparison test
- Limit comparison test

Infinite limits of integration.

Consider the area under the curve $y = e^{-x/2}$ in the 1st quadrant:



$$A = \int_0^\infty e^{-x/2}\, dx$$

$$= \lim_{b\to\infty} \int_0^b e^{-x/2}\, dx \qquad \begin{array}{l} u = -x/2 \Rightarrow \begin{array}{l} u(b) = -b/2 \\ u(0) = 0 \end{array} \\ du = -\frac{dx}{2} \Rightarrow dx = -2\, du \end{array}$$

$$= \lim_{b\to\infty} \int_0^{-b/2} -2e^u\, du$$

$$= \lim_{b\to\infty} \left[-2e^u \Big|_{u=0}^{-b/2} \right]$$

The area is finite!

$$= \lim_{b\to\infty} \left[-2e^{-b/2} + 2e^0 \right]$$

$$= \lim_{b\to\infty} \left[2 - 2e^{-b/2} \right]$$

$$= 2 - 2 \lim_{b\to\infty} e^{-b/2}$$

$$= 2 - \underbrace{2 \exp\Big(\underbrace{-\tfrac{1}{2} \underbrace{\lim_{b\to\infty} b}_{\to\infty}}_{\to -\infty}\Big)}_{\to 0}$$

$$\boxed{= 2.}$$

DEFINITION. Integrals with infinite limits of integration are called IMPROPER INTEGRALS OF TYPE I;

- $\displaystyle\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx$, if $f(x)$ is cts. on $[a,\infty)$
- $\displaystyle\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty}\int_a^b f(x)\,dx$, if $f(x)$ cts. on $(-\infty,b]$
- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{c} f(x)\,dx + \int_c^\infty f(x)\,dx$, if $f(x)$ cts. $(-\infty,\infty)$

where c is any real number.

If the limit is finite, we say the improper integral CONVERGES and is the VALUE of the improper integral.

Otherwise, the improper integral DIVERGES.

Example: 1, p. 505: Is the area under the curve $y=\dfrac{\ln(x)}{x^2}$ from $x=1$ to $x=\infty$ finite? If yes, what is its value?

$$A = \int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx$$

$u=\ln(x)$, $v=-\frac{1}{x}$

$du=\frac{1}{x}\,dx$, $dv=\frac{1}{x^2}\,dx$

$\int u\,dv = uv-\int v\,du$

$$= \lim_{b\to\infty}\left[-\frac{\ln(x)}{x}\Big|_{x=1}^{b} - \int_1^b\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx\right]$$

$\int_1^b \frac{1}{x^2}\,dx$

$$= \lim_{b\to\infty}\left[-\frac{\ln(x)}{x} - \frac{1}{x}\Big|_{x=1}^{b}\right] = \lim_{b\to\infty}\left[\frac{-\ln(x)-1}{x}\Big|_{x=1}^{b}\right]$$

$$= \lim_{b\to\infty}\left[\frac{-\ln(b)-1}{b} + \frac{\ln(1)+1}{1}\right] = \lim_{b\to\infty}\left[1-\frac{1}{b}-\frac{\ln(b)}{b}\right]$$

Example, ct'd.

$$A = \lim_{b\to\infty}\left[1 - \frac{1}{b} - \frac{\ln(b)}{b}\right]$$

$$= 1 - \lim_{b\to\infty}\frac{1}{b} - \underbrace{\lim_{b\to\infty}\frac{\ln(b)}{b}}_{\text{form } \text{"}\frac{\infty}{\infty}\text{"}}$$

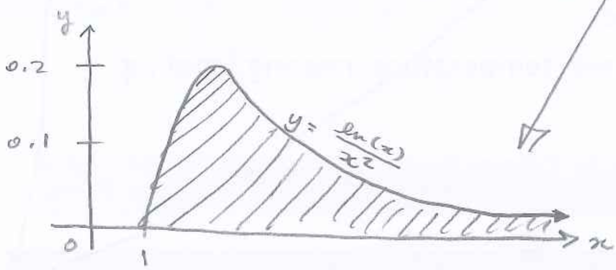
$f(b) = \ln(b)$, $g(b) = b$
$f'(b) = 1/b$, $g'(b) = 1$

$$\overset{?}{\underset{\text{l'Hôp}}{=}} 1 - \lim_{b\to\infty}\frac{1}{b} - \lim_{b\to\infty}\frac{1/b}{1}$$

$$= 1 - 2\lim_{b\to\infty}\frac{1}{b}.$$

$$= \boxed{1}$$

A



- The integral $\int_1^\infty \frac{dx}{x^p}$.

Example 2
p. 507

For which values of p does $\int_1^\infty \frac{dx}{x^p}$ converge? When it does converge, what is its value?

$$\int_1^\infty \frac{dx}{x^p} = \boxed{\lim_{b\to\infty}\int_1^b \frac{dx}{x^p}} = \lim_{b\to\infty}\left[\frac{x^{(1-p)}}{(1-p)}\Bigg|_{x=1}^{b}\right] \quad \leftarrow \text{IF } p \neq 1$$

$\lim_{c\to\infty}\frac{1}{\sqrt{c}} = \infty$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left[b^{1-p} - 1\right]$$

$\lim_{c\to\infty}\frac{1}{c^2} = 0$

$$= \frac{1}{p-1} + \frac{1}{1-p}\lim_{b\to\infty}\left[\frac{1}{b^{p-1}}\right]$$

Recall: $\lim_{c\to\infty}\frac{1}{c^k} = \begin{cases} 0, & k>1 \\ \infty, & k<1 \end{cases}$

$$= \begin{cases} \infty, & p \leq 1 \\ \frac{1}{p-1}, & p > 1 \end{cases}$$

and if $p=1$: $\boxed{\lim_{b\to\infty}\int_1^b \frac{dx}{x^p}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\left[\ln(x)\Big|_{x=1}^{b}\right]$

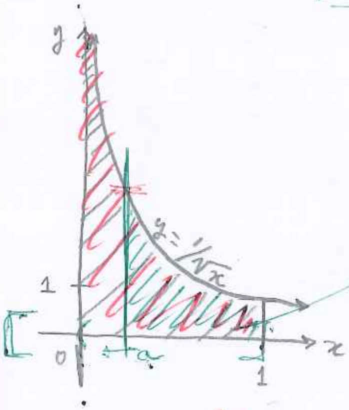
$$= \lim_{b\to\infty}\left[\ln(b)\right] = \infty.$$

• Integrands with vertical asymptotes:

Example 3

P. 507

Consider the region in the 1st quadrant that lies under the curve $\boxed{y = \frac{1}{\sqrt{x}}}$ from $x = 0$ to $x = 1$:



$$A = \int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} \left[\int_a^1 \frac{1}{\sqrt{x}}\, dx \right]$$

$$= \lim_{a \to 0^+} \left[2\sqrt{x} \Big|_{x=a}^{1} \right]$$

$$= \lim_{a \to 0^+} \left[2 - 2\sqrt{a} \right]$$

$$= 2 - 2 \lim_{a \to 0^+} \sqrt{a} \quad (\to 0)$$

$$= \boxed{2}$$

DEFINITION. Integrals of functions that become infinite at a point within the interval of integration are called <u>IMPROPER INTEGRALS OF TYPE II</u>:

• f is cts. on $(a, b]$ and discts. at a:

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx$$

• f is cts. on $[a, b)$ and discts. at b:

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx$$

• f is discts. at $c \in (a, b)$ and cts. on $[a, c) \cup (c, b]$:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

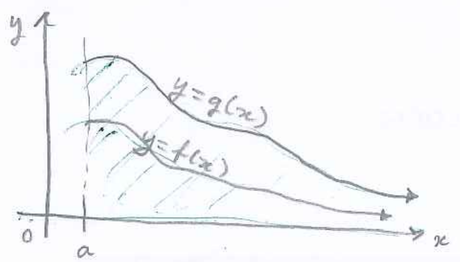
In each case, if the limit is finite we say the improper integral <u>CONVERGES</u> and its <u>VALUE</u> is the value of the limit; otherwise, we say it <u>DIVERGES</u>.

• Tests for convergence / divergence

Sometimes, we cannot evaluate an improper integral directly — and sometimes, the only thing we care about is whether the integral converges, not what it converges to.

THEOREM (Direct comparison test)

Let f and g be continuous functions on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then:

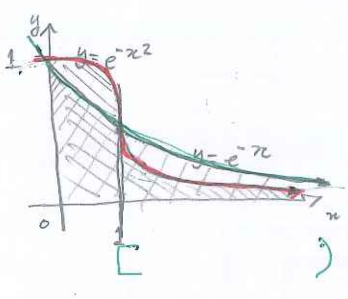


• IF $\int_a^\infty g(x)\,dx$ converges, THEN so does $\int_a^\infty f(x)\,dx$.

• IF $\int_a^\infty f(x)\,dx$ diverges, THEN so does $\int_a^\infty g(x)\,dx$.

NOTES:
• The "bottom" integral can force the "top" to DIVERGE
• The "top" integral can force the "bottom" to CONVERGE
• IT IS MEANINGLESS IF THE "TOP" INTEGRAL DIVERGES
• IT IS MEANINGLESS IF THE "BOTTOM" INTEGRAL CONVERGES

Example:



Interested in whether $\int_1^\infty e^{-x^2}\,dx$ converges.

Notice for all $x \ge 1$, $e^{-x^2} \le e^{-x}$

So if $\int_1^\infty e^{-x}\,dx$ converges, then so does our integral $\int_1^\infty e^{-x^2}\,dx$.

Let's see: $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_{x=1}^{b} = \lim_{b\to\infty}\left[\cancel{-e^{-b}} + e^{-1}\right] = \boxed{\frac{1}{e}}$.

So since $\int_1^\infty e^{-x}\,dx = \frac{1}{e}$ converges and $e^{-x^2} \le e^{-x}$ for $x \ge 1$, $\int_1^\infty e^{-x^2}\,dx$ conv. as well

THEOREM (Limit comparison test)

Let f, g be positive functions continuous on $[a, \infty)$.

IF $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where $\boxed{0 < L < \infty}$

THEN the integrals $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$

either both converge or both diverge.

Example. Show that $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\displaystyle\int_1^\infty \frac{dx}{x^2}$.
Find & compare the two integral values.

Let $f(x) := \dfrac{1}{1+x^2}$ and $g(x) := \dfrac{1}{x^2}$. Then both f and g are positive and cts. on $[1, \infty)$, so to use the limit comparison test:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{x^2}{1+x^2}\underbrace{\left(\frac{1/x^2}{1/x^2}\right)}_{1} = \lim_{x\to\infty} \frac{1}{\underbrace{\cancel{\tfrac{1}{x^2}}}_{0} + 1} = 1,$$

and $0 < 1 < \infty$, so by the limit comparison test, the integrals $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ either both converge, or both diverge. To find out which, we evaluate $\displaystyle\int_1^\infty \frac{dx}{x^2}$ directly:

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\Big|_{x=1}^{b} \right] = \lim_{b\to\infty} \left[-\underbrace{\cancel{\tfrac{1}{b}}}_{0} + 1 \right] = 1.$$

So, $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, and therefore $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ also converges.

To find the value of $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$, we use an alternative method (not discussed here), and see: $\boxed{\displaystyle\int_1^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}},$

The upshot: even if both converge, both may converge to different values!