Lecture 10: Lines & Planes in Space; Curves in Space & Motion

(12.5, 13.1 ~~[illegible]~~)

Announcements: • HW6 to be posted Weds. a.m., due next Weds

- Final Exam next Thursday (Aug. 20)
  - Remember deal with midterm (only for those who took the midterm in July): if you score higher on the final, that grade gets counted twice.
  - Review on Tuesday (August 18)

12.5: Lines and Planes in space.

In a plane, a line can be determined by

- two points
- a point, and a slope

In 3-space, a line can be determined by...

- two points
- a point, and a vector giving the line's direcᵢₙ

~~The~~ An eq'n of the line $\parallel$ to vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$

parallel

and passing through the point $P_0 := P_0(x_0, y_0, z_0)$ is:

$$x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = \left(x_0\,\hat{i} + y_0\,\hat{j} + z_0\,\hat{k}\right) + t\left(v_1\,\hat{i} + v_2\,\hat{j} + v_3\,\hat{k}\right)$$

where $t \in (-\infty, \infty)$.

L10, ct'd,

So: Three equations:

$$\left.\begin{array}{ll} (1) & x = x_0 + tv_1 \\ (2) & y = y_0 + tv_2 \\ (3) & z = z_0 + tv_3 \end{array}\right\} \text{ t must be the same!}$$

These 3 eq'ns constitute a parametrization.

EXAMPLE 1, p.733 Find a parametrization for the line through $(-2,0,4)$ parallel to $\vec{v} = \langle 2,4,-2\rangle$.

Use the formulas: $P_0(x_0, y_0, z_0) = (-2,0,4)$

$\vec{v} = \langle v_1, v_2, v_3\rangle = \langle 2,4,-2\rangle$

So (1) $x = x_0 + tv_1$

$x = -2 + t(2)$

(2) $y = y_0 + tv_2$

$y = 0 + t(4)$

(3) $z = z_0 + tv_3$

$z = 4 + t(-2)$

And the parametrization:

$$\boxed{\begin{array}{l} x = -2 + 2t \\ y = 4t \\ z = 4 - 2t. \end{array}}$$

$\boxed{\text{EXAMPLE 2, p. 733}}$ Find parametric eq'ns for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Want a vector $\parallel$ to the line, so choose $\overrightarrow{PQ}$:

$$\overrightarrow{PQ} = Q - P = \langle 1 - (-3), -1 - 2, 4 - (-3) \rangle$$
$$= \langle 4, -3, 7 \rangle.$$

Now, use $P_0 = P = P(-3, 2, -3)$ and $\vec{v} = \overrightarrow{PQ} = \langle 4, -3, 7 \rangle$

so:

$$\left.\begin{array}{ll} (1) & x = x_0 + t v_1 \\ (2) & y = y_0 + t v_2 \\ (3) & z = z_0 + t v_3 \end{array}\right\} \Rightarrow \boxed{\begin{array}{ll} (1) & x = -3 + t(4) \\ (2) & y = 2 + t(-3) \\ (3) & z = -3 + t(7) \end{array}}$$

Could also use $P_0 = Q(1, -1, 4)$ with $\vec{v} = \overrightarrow{PQ} = \langle 4, -3, 7 \rangle$:

$$\boxed{\begin{array}{ll} (1) & x = 1 + t(4) \\ (2) & y = -1 + t(-3) \\ (3) & z = 4 + t(7) \end{array}}$$

So parametric eq'ns are not unique (... but we knew that!).

EXAMPLE 3, p. 733

Parametrize the line segment joining the two points $P(-3,2,-3)$ and $Q(1,-1,4)$.

Use the eq'ns from before:

(1) $x = -3+4t$

(2) $y = 2-3t$

(3) $z = -3+7t$

And find the "starting" and "ending" t-values such that:

$$P(-3,2,-3) = (-3+4t_s,\ 2-3t_s,\ -3+7t_s)$$

and

$$Q(1,-1,4) = (-3+4t_e,\ 2-3t_e,\ -3+7t_e).$$

Observe: $t_s = 0$ gives the point $(-3,2,-3)$

and $t_e = 1$ gives the point $(1,-1,4)$.

So $t \in [0,1]$.

Think of $\vec{r}(t) = \vec{r_0} + t\vec{v}$ instead as:

$$\vec{r}(t) = \underbrace{\vec{r_0}}_{\text{initial pos'n}} + \underbrace{t}_{\text{time}}\ \underbrace{|\vec{v}|}_{\text{speed}}\ \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{direc'n (unit vector)}}$$

EXAMPLE 4, p. 734 A helicopter flies directly from a helipad at the origin in the direc'n of the point (1, 1, 1) at a speed of 60 ft/sec. What is the pos'n of the helicopter after 10 sec?

Initial pos'n: $(0,0,0) = \vec{r}_0$

Speed: 60 ft/sec

Direc'n: $\hat{u} = \dfrac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|} = \dfrac{\langle 1,1,1\rangle}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$

So

$$\vec{r}(t) = \vec{r}_0 + t\,(\text{speed})\,\hat{u}$$

$$= \vec{0} + t\left(60\ \text{ft/sec}\right)\left(\frac{1}{\sqrt{3}}\right)\langle 1,1,1\rangle$$

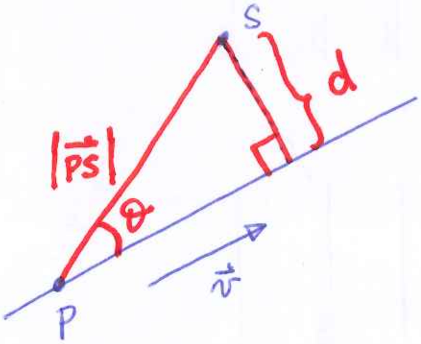
$$= t\left(20\sqrt{3}\ \text{ft/sec}\right)\langle 1,1,1\rangle.$$

At $t = 10$ sec:

$$\vec{r}(10\ \text{sec}) = (10\ \text{sec})(20\sqrt{3}\ \text{ft/sec})\langle 1,1,1\rangle$$

$$= \langle 200\sqrt{3},\ 200\sqrt{3},\ 200\sqrt{3}\rangle\ \text{ft}.$$

Distance from a pt. to a line in space:



Recall: S $\frac{O}{H}$ C $\frac{A}{H}$ T $\frac{O}{A}$

$$\sin(\theta) = \frac{d}{|\vec{PS}|}$$

so $d = |\vec{PS}| \sin(\theta)$.

Recall: $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta \, \hat{n}$

so $\vec{PS} \times \vec{v} = |\vec{PS}||\vec{v}| \sin\theta \, \hat{n}$

$\therefore$ $|\vec{PS} \times \vec{v}| = |\vec{PS}||\vec{v}||\sin\theta| \underbrace{|\hat{n}|}_{1}$

So $d = |\vec{PS}| \sin\theta$

$$\boxed{d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}}$$

L10, ct'd.

EXAMPLE 5, p. 734

Find the distance from $S(1,1,5)$ to the line

$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$

What is a point on L?

- Take $t=0$: $P(1,3,0)$

What vector is L parallel to?

- Take t-coefficients: $\vec{v} = \langle 1, -1, 2 \rangle$.

Now, use distance formula:

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

where $|\vec{v}| = |\langle 1,-1,2\rangle| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{6}$

and $\vec{PS} = S - P = \langle 1-1, 1-3, 5-0 \rangle = \langle 0, -2, 5 \rangle$.

Now, $\vec{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ -1 & 2 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} \hat{k}$

$= (-4+5)\hat{i} - (-5)\hat{j} + (2)\hat{k}$

$= \langle 1, +5, 2 \rangle$.

$$\frac{|\langle 1, +5, 2\rangle|}{} = \frac{\sqrt{1^2+(+5)^2+2^2}}{} = \frac{\sqrt{30}}{} \neq \sqrt{}$$

L10, ct'd.

Eq'n for a plane in space.

The plane through $P_0(x_0, y_0, z_0)$ normal to perpendicular the vector $\hat{n} = \langle A, B, C \rangle$ has:

not necessarily a unit vector!

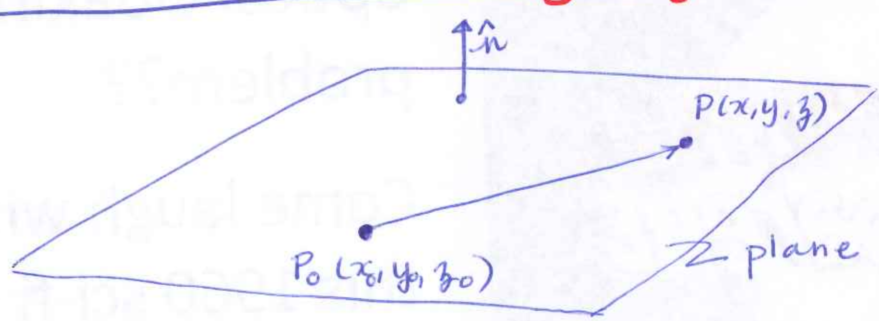
VECTOR EQ'N: $\hat{n} \cdot \overrightarrow{P_0P} = 0$, $P(x, y, z)$

EQUIVALENT TO: $\langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$

$Ax + By + Cz = Ax_0 + By_0 + Cz_0$

Why?



EXAMPLE 6, p. 735

Find an eq'n for the plane containing $P_0(-3, 0, 7)$ perpendicular to $\vec{n} = \langle 5, 2, -1 \rangle$.

The eq'n: $\vec{n} \cdot \overrightarrow{P_0P} = 0$

$\langle 5, 2, -1 \rangle \cdot \langle x - (-3), y - 0, z - 7 \rangle = 0$

$5(x + 3) + 2y - (z - 7) = 0$

$\boxed{5x + 2y - z = -22.}$

EXAMPLE 7, p. 736

Find an eq'n for the plane through 3 pts: $A(0,0,1)$, $B(2,0,0)$, $C(0,3,0)$.

We need a point and a vector normal to the plane.

Recall: A vector perpendicular to the plane containing $\vec{u}$ and $\vec{v}$ is $\vec{u} \times \vec{v}$.

So let's pick two vectors $\vec{AB}$ and $\vec{AC}$ and use them to find a suitable $\vec{n}$:

$$\vec{n} = \vec{AB} \times \vec{AC} = \langle 2-0, 0-0, 0-1 \rangle \times \langle 0-0, 3-0, 0-1 \rangle$$

$$= \langle 2, 0, -1 \rangle \times \langle 0, 3, -1 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \hat{k}$$

$$= (0-(-3))\hat{i} - (-2-0)\hat{j} + (6-0)\hat{k}$$

$$= \langle 3, 2, 6 \rangle.$$

So use this $\vec{n} = \langle 3, 2, 6 \rangle$ with the point $A(0,0,1)$:

$$\vec{n} \cdot \vec{AP} = 0 \quad \Rightarrow \quad 3(x-0) + 2(y-0) + 6(z-1) = 0$$

$\boxed{\text{EXAMPLE 7, ct'd}}$

Another way:

Choose points / vectors from among $\underset{=(0,0,1)}{A}$, $\underset{=(2,0,0)}{B}$, and $C\,(0,3,0)$ differently:

$$\vec{n} = \underset{AB}{\cancel{\overrightarrow{BA}}} \times \overrightarrow{CB} = \langle 2-0, 0-0, 0-1\rangle \times \langle 2-0, 0-3, 0-0\rangle$$

$$= \langle 2, 0, -1\rangle \times \langle 2, -3, 0\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 2 & -3 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 \\ -3 & 0 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 0 \\ 2 & -3 \end{vmatrix}\hat{k}$$

$$= (0-3)\hat{i} - (0+2)\hat{j} + (-6-0)\hat{k}$$

$$= \langle -3, -2, -6\rangle$$

$P_0 = B$ gives:

$$\vec{n} \cdot \overrightarrow{BP} = 0 \Rightarrow \langle -3, -2, -6\rangle \cdot \langle x-2, y-0, z-0\rangle = 0$$

$$\Rightarrow -3(x-2) - 2y - 6z = 0$$

$$-3x - 2y - 6z = -6$$

or $\quad 3x + 2y + 6z = 6$, as before.

L10, ct'd.

Two planes are parallel if their <u>normals</u> are $\parallel$.

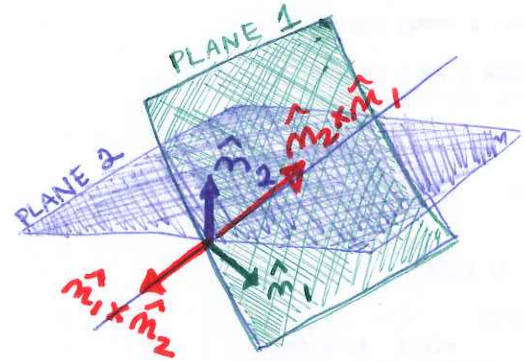
Any two planes that are <u>not</u> $\parallel$ intersect in a line.

(Any two lines that are not $\parallel$ intersect in a point.)

EXAMPLE 8, p. 736

Find a vector $\parallel$ to the line of intersec'n of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Well, the line ~~of~~ of intersec'n is $\perp$ to both $\hat{n}_1$ and $\hat{n}_2$ – therefore it is $\parallel$ to $\hat{n}_1 \times \hat{n}_2$.



So, find $\underbrace{\hat{n}_1}$ and $\underbrace{\hat{n}_2}$:

not necessarily unit vectors

$3x - 6y - 2z = 15 \quad : \quad \langle 3, -6, -2 \rangle$

$2x + y - 2z = 5 \quad : \quad \langle 2, 1, -2 \rangle$

And $$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -6 & -2 \\ 1 & -2 \end{vmatrix} \hat{i} - \begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} \hat{j} + \begin{vmatrix} 3 & -6 \\ 2 & 1 \end{vmatrix} \hat{k}$$

$$= (12 + 2)\hat{i} - (-6 + 4)\hat{j} + (3 + 12)\hat{k}$$

$$= \langle 14, 2, 15 \rangle.$$

So $\langle 14, 2, 15 \rangle$ is a parallel to the line of intersec'n.

L10, ct'd.

EXAMPLE 9, p. 736 Find parametric eq'ns for the line in which the planes $3x-6y-2z=15$ and $2x+y-2z=5$ intersect.

Well, we need a point on the line, and we need a vector $\parallel$ to the line.

We have the vector from before: $\langle 14, 2, 15\rangle$.

To find a point $P(x,y,z)$ that satisfies BOTH

$$3x-6y-2z=15 \quad \text{and}$$
$$2x+y-2z=5,$$

we do something arbitrary with one variable — like, e.g., setting $x=0$ — and solve for the other two:

$$x=0 \Rightarrow \quad -6y-2z=15$$
$$y-2z=5.$$

Subtract:
$$\begin{array}{r} -6y-2z=15 \\ -\ (y-2z=5) \\ \hline -7y=10 \end{array} \Rightarrow y=-10/7$$

and from (2), $z=-\frac{5}{2}+\frac{1}{2}y=-\frac{5}{2}-\frac{10}{14}=\frac{-45}{14}.$

Point is $P(0, -10/7, -45/14)$

So the eq'ns are:

$$\boxed{x=0+14t, \quad y=-\frac{10}{7}+2t, \quad z=\frac{-45}{14}+15t.}$$

$\boxed{\text{EXAMPLE 10, p. 737}}$ Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Well, a point on the line is

$$\left(\frac{8}{3} + 2t, \ -2t, \ 1+t\right) =: (x, y, z)$$

and we want to find the t-value that makes this point (x, y, z) satisfy the eq'n $3x + 2y + 6z = 6$.

Substitute:

if and only if

$$3x + 2y + 6z = 6$$

$$\Leftrightarrow \quad 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$$

$$\Leftrightarrow \quad 8 + 6t - 4t + \cancel{6} + 6t = \cancel{6}$$

$$\Leftrightarrow \quad 8 + (6 - 4 + 6)t = 0$$

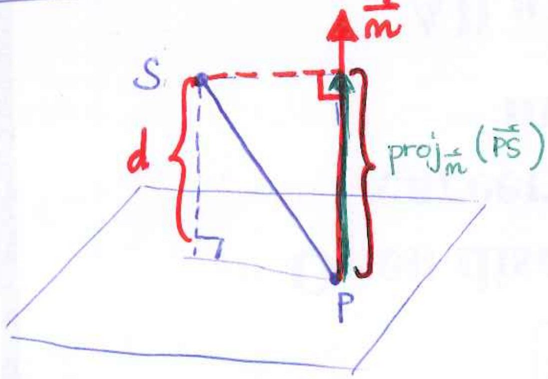
$$\Leftrightarrow \quad 8t = -8$$

$$\Leftrightarrow \quad t = -1.$$

So ~~[scribbled]~~ our point is given by $t = -1$, or:

$$(x, y, z) = \left(\frac{8}{3} + 2(-1), \ -2(-1), \ 1 + (-1)\right) = \boxed{\left(\frac{2}{3}, 2, 0\right)}$$

The distance from a point to a plane.



$$d = \left|\, \mathrm{proj}_{\vec{m}}(\vec{PS}) \,\right| \quad \text{mag.}$$

$$= \left|\, \vec{PS} \cdot \frac{\vec{m}}{|\vec{m}|} \,\right| \quad \text{abs. val}$$

EXAMPLE
11, p. 737

Find the distance from $S(1,1,3)$ to the plane $3x + 2y + 6z = 6$.

Well, we want a point P on the plane, and we want a vector $\vec{m}$ normal to the plane:

$\vec{m} = \langle 3, 2, 6 \rangle$ and $\underbrace{P(0,0,1)}$ do the job.

could be any $P(x,y,z)$ satisfying the eq'n $3x+2y+6z=6$. For example, $P(0,3,0)$ and $P(2,0,0)$ wo

So $d = \left| \vec{PS} \cdot \frac{\vec{m}}{|\vec{m}|} \right| = \frac{1}{|\vec{m}|} \left| \vec{PS} \cdot \vec{m} \right| = \left(\frac{1}{\sqrt{3^2+2^2+6^2}} \right) \left| \langle 1-0, 1-0, 3-1 \rangle \cdot \langle 3,2,6$

$$= \left(\frac{1}{\sqrt{9+4+36}} \right) \left| \langle 1,1,2 \rangle \cdot \langle 3,2,6 \rangle \right| = \left(\frac{1}{\sqrt{49}} \right) \left| 1 \cdot 3 + 1 \cdot 2 + 2 \cdot \right.$$

$$= \left(\frac{1}{7} \right) \left| 3 + 2 + 12 \right| = \frac{|17|}{7} = \boxed{\frac{17}{7}}.$$

L10, ct'd.

13.1 : Curves in Space and their tangents.

The parametrized curve

$$x = f(t), \quad y = g(t), \quad z = h(t), \qquad t \in I$$

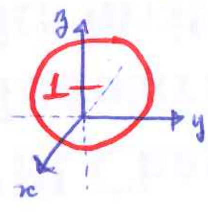
may define the path of a particle through space; can also represent in vector form:

$$\vec{r}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}$$

$$= \langle f(t), g(t), h(t) \rangle .$$

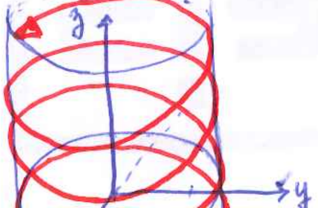
$\vec{r}(t)$ is said t/b a VECTOR-VALUED FUNCTION with the COMPONENT FUNCTIONS $f(t)$, $g(t)$, and $h(t)$, and it has domain I.

$\boxed{\text{EXAMPLE 1, p. 752}}$ The vector fn. $\vec{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$ is familiar ...



$(t \in (-\infty, \infty))$

But what about $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$?



- Still lies on cylinder $x^2 + y^2 = 1$
- curve rises as $\hat{k}$ component increases

Limits & continuity

DEF. $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$ IFF $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall t \in D$ (fn's domain)

$\underbrace{|t - t_0|}_{\text{abs. value}} < \delta \Rightarrow \underbrace{|\vec{r}(t) - \vec{L}|}_{\text{vector magnitude}} < \varepsilon$

- This is similar to the scalar-valued fn. version:

$$\lim_{x \to x_0} f(x) = L \text{ iff } \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in D,$$
$$|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

- To compute the limit of a vector-valued fn., we compute the limits of its components!

$$\boxed{\lim_{t \to t_0} \left[\langle f(t), g(t), h(t) \rangle \right] = \left\langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right\rangle}$$

DEF. A vector fn. $\vec{r}(t)$ is CONTINUOUS at $t = t_0$ if $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$, and it is just "CONTINUOUS" if it is cts. at all points in its domain.

EXAMPLE 2, p. 753 $\lim_{t \to \frac{\pi}{4}} \langle \cos(t), \sin(t), t \rangle = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}, \frac{\pi}{4} \rangle$

DEF. $\vec{r}\,'(t) = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}\langle f(t), g(t), h(t)\rangle$

$= \left\langle \dfrac{df}{dt}, \dfrac{dg}{dt}, \dfrac{dh}{dt} \right\rangle$

proof on p. 754

DEF. If $\vec{r}(t)$ is the pos'n vector of a particle moving along a smooth curve in space, then:

continuously diff'ble and $\vec{r}\,'(t) \neq \vec{0}$

(1) VELOCITY: $\vec{v} = \dfrac{d\vec{r}}{dt}$

(2) SPEED: $s = |\vec{v}|$

(3) ACCELERAT'N: $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$

(4) DIREC'N OF MOTION: $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|}$

EXAMPLE 4, p. 755 $\quad \vec{r}(t) = \langle 2\cos t, 2\sin t, 5\cos^2 t\rangle$.

$\vec{v}(t) = \langle -2\sin t, 2\cos t, -10\cos(t)\sin t\rangle$

$s(t) = \sqrt{4\sin^2 t + 4\cos^2 t + 100\cos^2\sin^2 t} = \sqrt{4 + 100\cos^2 t\sin^2 t}$

$\vec{a}(t) = \langle -2\cos t, -2\sin t, -10\cos^2 t + 10\sin^2 t\rangle$

Differentiat'n rules.

(1) $\frac{d}{dt}\vec{c} = \vec{0}$ ($\vec{c}$ is a constant vector)

c is a const. scalar

(2) $\frac{d}{dt}[c\,\vec{u}(t)] = c\,\frac{d\vec{u}}{dt}$ (constant multiple)

(3) $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt}$ (sum)

(4) $\frac{d}{dt}[\vec{u}(t) - \vec{v}(t)] = \frac{d\vec{u}}{dt} - \frac{d\vec{v}}{dt}$ (diff.)

(5) $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \frac{d\vec{u}}{dt} \cdot \vec{v}(t) + \vec{u}(t) \cdot \frac{d\vec{v}}{dt}$

(6) $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \frac{d\vec{u}}{dt} \times \vec{v}(t) + \vec{u}(t) \times \frac{d\vec{v}}{dt}$

(7) $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\,\frac{d}{dt}[\vec{u}]\Big|_{f(t)}$ (chain)

Preserve the order in the cross product — it is <u>not</u> commutative!

rule proofs on p. 756.