Lecture 11: Projectile Motion, Arc Length, Curvature (13.2–13.4)

Announcements:

- FINAL EXAM next Thursday, Aug 20
  - Class on Tuesday will be a review
  - See syllabus for chapters/sections covered (11.1–5, 12.1–5, 13.1–4)
    - NO SEC'N 13.5 (perhaps a bonus question?)

- This is the last lecture with new material
  - Tuesday will be review
  - Please ask questions and/or let me know what to cover in this session if you're having troubles

- HW6 posted yesterday, due Wednesday (Aug 19) at 11:59 p.m.

13.2: Integrals of vector fns. & projectile motion.

DEF. $$\int_a^b \langle f(t), g(t), h(t) \rangle \, dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

EXAMPLE 2, p.760

$$\int_0^{\pi} \langle \cos(t), 1, -2t \rangle \, dt =$$

$$= \left\langle \int_0^{\pi} \cos t \, dt, \int_0^{\pi} 1 \, dt, \int_0^{\pi} -2t \, dt \right\rangle$$

$$= \left\langle \sin t \Big|_0^{\pi}, t \Big|_0^{\pi}, -t^2 \Big|_0^{\pi} \right\rangle$$

$$= \left\langle \sin\pi - \sin(0), \pi - 0, -\pi^2 - 0^2 \right\rangle$$

$$= \left\langle 0, \pi, -\pi^2 \right\rangle.$$

DEF. The vector eq'n of the path of ideal projectile motion is:

initial ~~velocity~~ speed

firing angle

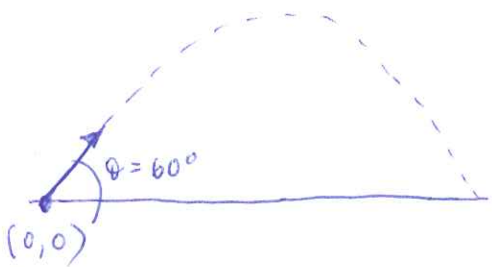
gravity

$$\vec{r}(t) = \left\langle v_0 (\cos\alpha) t, \; v_0 (\sin\alpha) t - \frac{1}{2} g t^2 \right\rangle$$

x = distance downrange, y = height

EXAMPLE
4, p. 762

A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec & launch angle 60°. Where will the projectile be 10 sec. later?



Eq'n for ideal proj. motion:

$$\vec{r}(t) = (v_0 \cos\alpha)t\,\hat{i} + \left((v_0 \sin\alpha)t - \tfrac{1}{2}gt^2\right)\hat{j}.$$

Here, $v_0 =$

$\alpha =$

So the eq'n $\vec{r}(t) =$

We want to find $\vec{r}(10\text{ sec})$.

$\vec{r}(10\text{ sec}) =$

For ideal projectile motion:

Max height: $y_{max} = \dfrac{(v_0 \sin\alpha)^2}{2g}$

Flight time: $t = \dfrac{2 v_0 \sin\alpha}{g}$

Range: $R = \dfrac{v_0^2}{g} \sin 2\alpha$

Notice: $\begin{cases} x = (v_0 \cos\alpha)t \\ y = (v_0 \sin\alpha)t - \frac{1}{2} g t^2 \end{cases}$

implies $t = \dfrac{x}{v_0 \cos\alpha}$, so $y = \dfrac{v_0 \sin\alpha}{v_0 \cos\alpha} x - \dfrac{g x^2}{2(v_0^2 \cos^2\alpha)}$

$= (\tan\alpha) x - \left(\dfrac{g}{2 v_0^2 \cos^2\alpha}\right) x^2.$

This is the eq'n of a ________

EXAMPLE
5, p.763

A baseball is hit when it is 3ft. above the ground. It leaves the bat with the initial speed of 152 ft/sec, making an angle of 20° with horizontal. At the instant the ball is hit, a gust of wind blows in the horizontal direc'n directly opposite the direc'n the ball takes toward the out field, adding a component of $-8.8\,\hat{i}$ ft/sec to the ball's initial velocity (8.8 ft/sec = 6 mph).

(a) Find a position vector for the path of the ball.

(b) How high does the baseball go; & when does it reach maximum height?

(c) Assuming the ball is not caught, find its range and flight time.

EXAMPLE 5, p. 763 ct'd.

(a) Notice that when a projectile is launched from a point (x_0, y_0) instead of from the origin, this just adds $\langle x_0, y_0 \rangle$ to the pos'n vector:

$$\vec{r}(t) = \left\langle x_0 + (v_0 \cos\alpha)t \quad , \quad y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \right\rangle .$$

Notice also that

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left\langle v_0 \cos\alpha \quad , \quad v_0 \sin\alpha - gt \right\rangle \Rightarrow \vec{v_0} = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \langle 0, -g \rangle .$$

When we go "backward" and integrate $\vec{a}$ to find $\vec{v}$:

$$\vec{v}(t) = \int_0^t \vec{a}(\tau)\, d\tau = \int_0^t \langle 0, -g \rangle\, d\tau = \langle 0, -g\tau \rangle \Big|_0^t =$$

$$= \langle 0, -gt \rangle + \vec{v_0}$$

and for us, $\vec{v_0} = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle + \overbrace{\langle -8.8, 0 \rangle}^{\text{the wind gust}}$

so $\vec{v}(t) = \langle v_0 \cos\alpha , v_0 \sin\alpha - gt \rangle + \langle -8.8, 0 \rangle$

$$= \langle v_0 \cos\alpha - 8.8 , \quad v_0 \sin\alpha - gt \rangle$$

→

$\boxed{\text{EX. 5, ct'd} \quad \text{p. 763}}$

So $\vec{r}(t) = \int_0^t \vec{v}(\tau)\, d\tau$

$= \int_0^t \langle 0, -g\tau \rangle + \vec{v}_0 \, d\tau$

$= \langle 0, -\frac{1}{2}g\tau^2 \rangle + \vec{v}_0 \tau \Big|_0^t$

$= \langle 0, -\frac{1}{2}gt^2 \rangle + \vec{v}_0 t + \vec{r}_0$

And for us, $\vec{r}(t) = \langle 0, -\frac{1}{2}gt^2 \rangle + \overbrace{\langle -8.8, 0 \rangle}^{\text{wind gust}} t + \overbrace{\langle 0, 3 \rangle}^{\text{initial pos'n}}$

$+ \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$

$= \langle (v_0 \cos\alpha - 8.8)t, (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3 \rangle$

and $v_0 = 152$ ft/sec, $\alpha = 20°$. So

$$\boxed{\vec{r}(t) = \langle (152\cos(20°) - 8.8)t, (152\sin(20°))t - \tfrac{1}{2}gt^2 + 3 \rangle}$$

(b) What is max. height, & when achieved?

Set $\frac{dy}{dt} = 0$, i.e., $152\sin(20°) - gt = 0$

$t = \frac{152\sin(20°)}{g} \approx 1.62$ sec

$g \approx 32$ ft/sec²

This is a critical point – but $\frac{d^2y}{dt^2} = -g < 0$, so a max here, i.e., $\vec{r}(1.62 \text{ sec}) \approx \langle \_\_, 45.2 \text{ ft} \rangle$ is max height.

$\boxed{\text{EX 5, ct'd p. 763}}$ (c) When the baseball lands? where?

Want $\vec{r}(t)\Big|_{y\text{-comp.}} = 0$, i.e., $(152 \sin 20^\circ)t - \frac{1}{2}gt^2 + 3 = 0$

$$\left(-\frac{1}{2}g\right)t^2 + (152 \sin 20^\circ)\,t + 3 = 0$$

$$t = \frac{-152 \sin 20^\circ \pm \sqrt{152^2 \sin^2(20^\circ) - 4\left(-\frac{1}{2}g\right)(3)}}{2\left(-\frac{1}{2}g\right)}$$

$$= \frac{152 \sin(20^\circ)}{g} \pm \frac{1}{g}\sqrt{152^2 \sin^2(20^\circ) + 6g}$$

So $t \approx -0.06$ sec and $t \approx 3.3$ sec satisfy this.

Only one is realistic — take $t \approx 3.3$ sec.

Then $\vec{r}(3.3 \text{ sec}) = \langle (152 \cos 20^\circ - 8.8)(3.3 \text{ sec}),\ 0 \rangle$

$$\approx \langle 442 \text{ ft},\ 0 \text{ ft} \rangle$$

So lands at 442 ft after 3.3 sec.

13.3: Arc Length in space.

Recall 2D formula: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

In 3D: The length of a smooth curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$, that is traced exactly once as t increases from a to b, is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt,$$

or equivalently,

$$\boxed{L = \int_a^b \sqrt{|\vec{v}(t)|}\, dt.}$$

$\boxed{\text{EXAMPLE } 1, \text{ p. } 769}$ A glider is soaring upward along the helix

$$\vec{r}(t) = \langle \cos t \,,\; \sin t \,,\; t \rangle .$$

How long is the glider's path from $t = 0$ to $t = 2\pi$?

Directed distance / arc length parameter.

Suppose $P(t_0) := (x(t_0), y(t_0), z(t_0))$ lies on the path of a particle whose pos'n is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Then the directed distance along the path from $P(t_0)$ to $P(t) := (x(t), y(t), z(t))$ is:

$$\boxed{s(t) = \int_{t_0}^{t} |\vec{v}(\tau)| \, d\tau}$$

- τ – "dummy variable", chosen just because t already appears in the upper limit.

EXAMPLE 2, p.769 The helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from before; take $t_0 = 0$, so $P(t_0) = (1, 0, 0)$. Then the distance the glider has travelled from $t_0 = 0$ to time $t > 0$ is

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)| \, d\tau = \int_{0}^{t} |\langle -\sin\tau, \cos\tau, 1 \rangle| \, d\tau$$

$$= \int_0^t \sqrt{2} \, d\tau = \boxed{t\sqrt{2}.}$$

Speed on a smooth curve.

If $\quad s(t) = \displaystyle\int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$, then apply the FTC :

$$\frac{ds}{dt} = |\vec{v}(t)| .$$

But we knew that !

Unit tangent vector.

Know $\vec{v}(t) := \dfrac{d\vec{r}}{dt}$ is a tangent vector to $\vec{r}(t)$.

Make it a unit vector:

$$\vec{T} := \frac{\vec{v}}{|\vec{v}|}$$

is a unit vector tangent to the curve.

$\boxed{\text{EXAMPLE 3, p. 770}}$ Find the unit tangent vector to $\vec{r}(t) = \langle 1 + 3\cos t,$
$3\sin t,\ t^2 \rangle$.

Well, $\vec{v}(t) = \langle$

$|\vec{v}(t)| = |\langle$

So $\quad \vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle}{\sqrt{}} = \langle$

13.4: Curvature & Normal vectors of a curve.

Recall: unit tangent vector $\vec{T} := \dfrac{\vec{v}}{|\vec{v}|}$,

and speed on a smooth curve is $\dfrac{ds}{dt} = |\vec{v}|$.

Since $\dfrac{ds}{dt} > 0$, s is strictly increasing (why is $|\vec{v}| = 0$ impossible?), so is therefore injective (one-to-one). So it has an inverse, and the derivative of that inverse is

$$\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\vec{v}|}.$$

Now also, $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$, so finally, we plug these into $\vec{T}$:

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{r}}{ds}.$$

As a particle moves along a smooth curve, $\vec{T} = \dfrac{d\vec{r}}{ds}$ turns as the curve bends (though is always a unit vector). The RATE AT WHICH $\vec{T}$ TURNS is the

CURVATURE : $\boxed{\kappa := \left| \dfrac{d\vec{T}}{ds} \right|}$

If $\left|\frac{d\vec{T}}{ds}\right|$ is large, $\vec{T}$ turns sharply as the particle passes through point P (i.e., the curvature is large).

Computing κ as $\left|\frac{d\vec{T}}{ds}\right|$ may be difficult, if the curve is not parametrized in terms of s. If we have it in terms of t (most common), then:

$$\boxed{\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|, \quad \text{where } \vec{T} = \frac{\vec{v}}{|\vec{v}|}}$$

[Why? — Chain rule: $\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\frac{dt}{ds} = \frac{d\vec{T}}{dt}\left(\frac{1}{|\vec{v}|}\right)$.]

EXAMPLE 1, p. 773 A straight line is parametrized $\vec{r}(t) = \vec{C} + t\vec{v}$ for constant vectors $\vec{C}$ and $\vec{v}$.

So $\vec{r}\,'(t) = \vec{v}(t) = \vec{v}$ is constant, so $\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|}$ is also constant — i.e., $\frac{d\vec{T}}{dt} = \vec{0}$.

Therefore, $\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{|\vec{v}|}|\vec{0}| = 0$.

THIS IS WHAT WE EXPECT!

EXAMPLE 2, p. 773

Find the curvature of a circle

$$\vec{r}(t) = \langle a\cos t,\ a\sin t \rangle .$$