ADMINISTRATIVE.

- HW 1 POSTED, DUE MONDAY 11:59 p.m..
- ALL WRITTEN HW NOW DUE MONDAYS 11:59 p.m.
- MyMathLab ASSIGNMENT 1 DUE <u>TONIGHT</u>, 11:59 p.m.
- OTHER MYMATHLABS STILL ON THE WEDNESDAY - FRI. SCHEDULE.

Lecture 2: Sequences (Sec. 10.1)

DEF. A SEQUENCE is an ordered list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

The numbers in the sequence are called TERMS of the sequence, e.g.:

$$\underbrace{2}_{\downarrow \text{ the 1st term}}, \underbrace{4}_{\searrow \text{ the 2nd term}}, 6, 8, 10, \ldots, \underbrace{2n}_{\searrow \text{ the } n^{th} \text{ term}}, \ldots$$

... etc. ...

For a term a_n in a sequence, the integer n is called the INDEX of a_n.

An INFINITE SEQUENCE can be thought of as a function whose domain is the set of positive integers, e.g.,

$$2, 4, 6, 8, 10, \ldots, 2n, \ldots$$

can be described by the formula $a_n = 2n$.

HOW TO DESCRIBE SEQUENCES?

- Listing terms: $\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$
  $\{b_n\} = \{-1, 2, -3, \ldots, n(-1)^n, \ldots\}$
- Writing a rule: $a_n = \sqrt{n}$, or $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$
  $b_n = n(-1)^n$, or $\{b_n\} = \{n(-1)^n\}_{n=1}^{\infty}$

L2, ct'd.

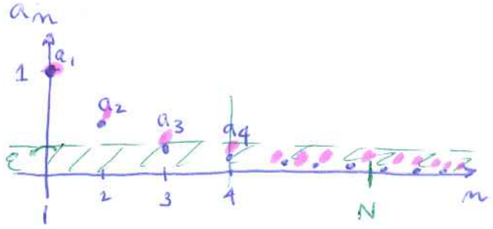
Compare the sequences:

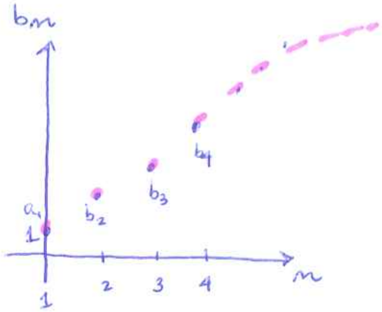
$$\{a_n\} := \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots \right\},$$

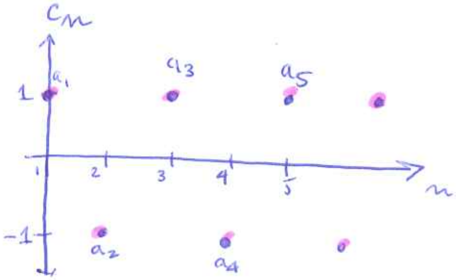
$$\{b_n\} := \left\{ \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{n}, \dots \right\}, \text{ and}$$

$$\{c_n\} := \left\{ 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots \right\}.$$

Could graph them:







Notice behavior as $n \to \infty$:

- $\{a_n\}$ terms approach 0
- $\{b_n\}$ terms approach ∞
- $\{c_n\}$ terms cycle between -1 and 1

We say $\{a_n\}$ <u>converges to 0</u>, and we write

$$\lim_{n \to \infty} a_n = 0, \quad \text{or} \quad \{a_n\} \to 0.$$

We say $\{b_n\}$ and $\{c_n\}$ <u>diverge</u>.

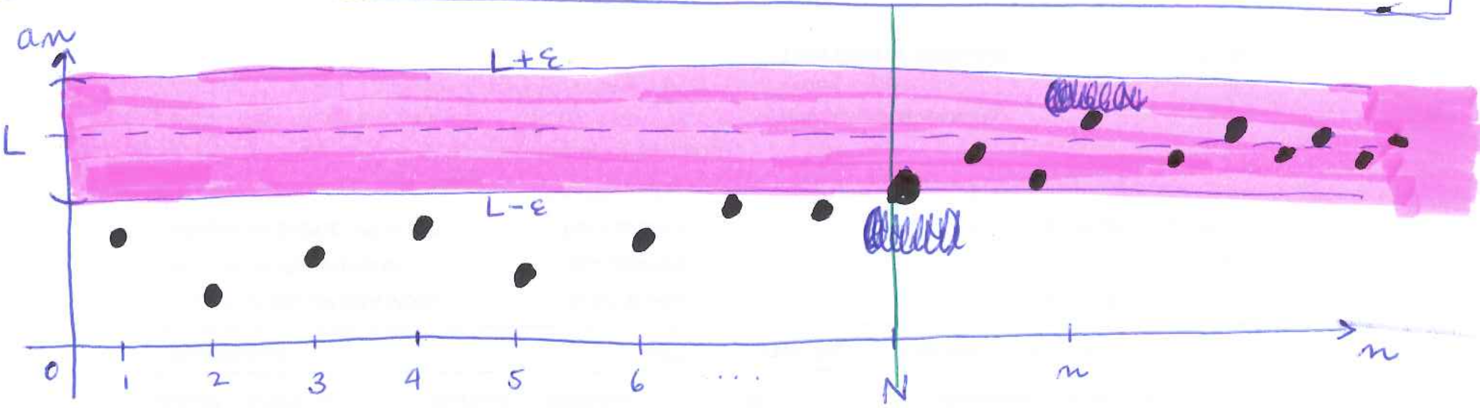
DEF. The sequence $\{a_n\}$ CONVERGES to the number L if:

For all $\varepsilon > 0$, there exists an integer N such that for all $n > N$, $|a_n - L| < \varepsilon$.

$$[\forall \varepsilon > 0,\ \exists N \in \mathbb{Z} \text{ s.t. } (\forall n \in \mathbb{Z},\ n > N \Rightarrow |a_n - L| < \varepsilon)].$$

If a_n converges to L, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, and L is called the LIMIT of the sequence.

If no such number L exists, then we say that the sequence $\{a_n\}$ DIVERGES.



EXAMPLE.
1(a), p. 574

Show that $\lim_{n\to\infty} \left(\frac{1}{n}\right) = \boxed{0}$

Let $\varepsilon > 0$ be given. Want to show that there exists an integer N such that $n > N$ implies $\left|\frac{1}{n} - 0\right| < \varepsilon$.

Scratch work: $\left|\frac{1}{n} - 0\right| < \varepsilon$

$$\Leftrightarrow \left|\frac{1}{n}\right| < \varepsilon$$

$$\Leftrightarrow \frac{1}{n} < \varepsilon \quad \text{since } n \in \mathbb{N}$$

$$\Leftrightarrow \boxed{n > \frac{1}{\varepsilon}} \leftarrow \text{STRATEGY}$$

Final proof: Let $\varepsilon > 0$ be given. Then if N is any integer greater than $\frac{1}{\varepsilon}$, we will have for all $n > N$ that $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| < \left|\frac{1}{N}\right| < \varepsilon$. This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Application: Let's choose $\varepsilon = 0.00001 = 1 \times 10^{-5} = \frac{1}{100{,}000}$.

Then we know that if $n > N$, where N is any integer greater than $\frac{1}{\varepsilon} = 100{,}000 = 10^5$, then $a_n = \frac{1}{n}$ is within ε of 0, i.e., $\frac{1}{n} < \varepsilon$.

DEF. The sequence $\{a_n\}$ DIVERGES TO INFINITY if for every number M there is an integer N such that for all n larger than N, $a_n > M$.

$[\forall M \in \mathbb{R}, \exists N \in \mathbb{Z} \text{ s.t. } (\forall n > N, a_n > M)]$.

If this holds true, we write $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$.

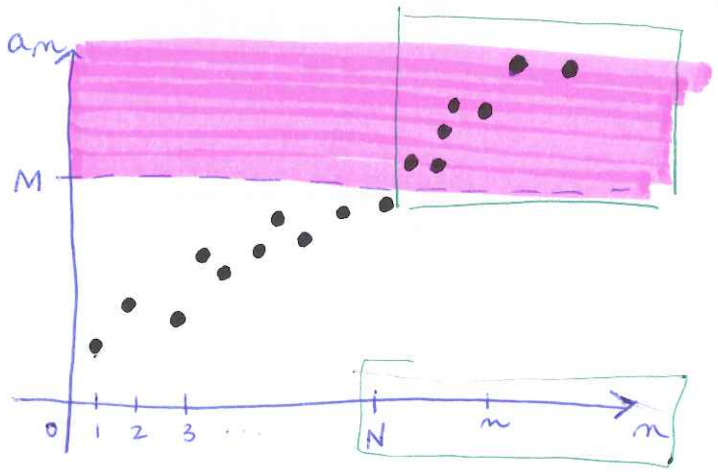
The sequence $\{a_n\}$ DIVERGES TO NEGATIVE INFINITY if for every number M there is an integer N such that for all n larger than N, $a_n < M$.

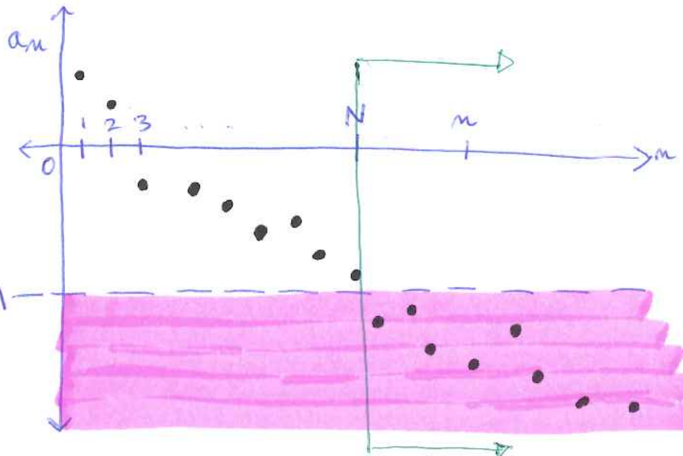
$[\forall M \in \mathbb{R}, \exists N \in \mathbb{Z} \text{ s.t. } (\forall n > N, a_n < M)]$.

If this holds true, we write $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$.

Note: • A sequence may diverge without diverging to ± ∞.
We've already seen an example:

$$\{b_n\} := \{1, -1, 1, -1, \dots\}$$





L2, ct'd.

Calculating limits of sequences.

THM 1, p. 576 Let $\{a_n\}$, $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers s.t. $a_n \to A$, $b_n \to B$.

① SUM RULE: $\lim\limits_{n\to\infty}(a_n + b_n) = A + B$

② DIFFERENCE RULE: $\lim\limits_{n\to\infty}(a_n - b_n) = A - B$

③ CONSTANT MULTIPLE: $\lim\limits_{n\to\infty}(k\cdot b_n) = kB$ (any # k)

④ PRODUCT RULE: $\lim\limits_{n\to\infty}(a_n \cdot b_n) = A\cdot B$

⑤ QUOTIENT RULE: $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$, provided $B \neq 0$.

Note: If we think of a sequence as a function whose domain is $\mathbb{N}$, then these rules — which are exactly parallel to those for limits of functions — should not be surprising.

$$\boxed{\lim_{n\to\infty}\frac{1}{n} = 0}$$

EXAMPLES. (a) $\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = (-1)\cdot\lim\limits_{n\to\infty}\frac{1}{n} = -1\cdot 0 = 0$

(b) $\lim\limits_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim\limits_{n\to\infty}(1) - \lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = 1-0$
$= 1$

(c) $\lim\limits_{n\to\infty}\left(\frac{5}{n^2}\right) = 5\cdot\left(\lim\limits_{n\to\infty}\frac{1}{n}\right)\cdot\left(\lim\limits_{n\to\infty}\frac{1}{n}\right) = 5\cdot 0\cdot 0 = 0$

(d) $\lim\limits_{n\to\infty}\dfrac{4-7n^6}{n^6+3}\dfrac{(1/n^6)}{(1/n^6)} \quad \lim\limits_{n\to\infty}\dfrac{(4/n^6)-7}{1+(3/n^6)} = \dfrac{0-7}{1+0} = -7$

Note: The converses of the rules in Theorem 1 often do NOT hold, e.g.:

$\{a_n\} = \{1, 2, 3, \dots\}$ and

$\{b_n\} = \{-1, -2, -3, \dots\}$ both diverge ($a_n \to \infty$, $b_n \to -\infty$)

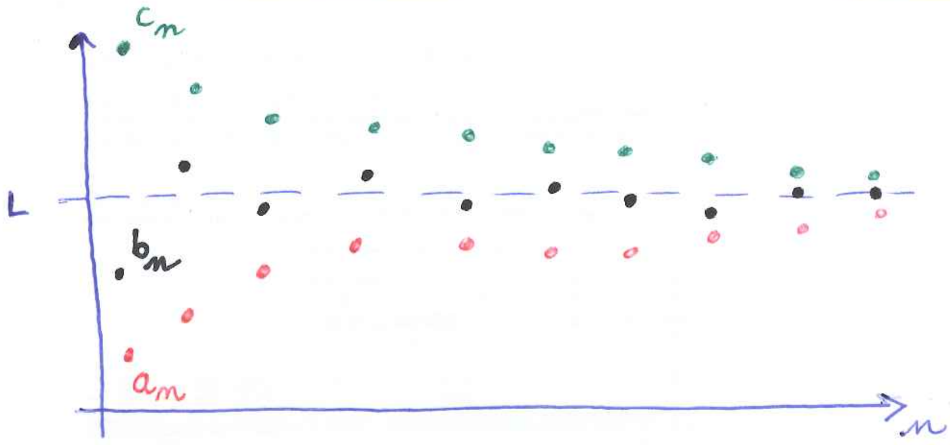
but their sum $\{a_n + b_n\} = \{0, 0, \dots, 0\}$ converges (to 0).

THM. ("SANDWICH THEOREM FOR SEQUENCES")

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences of real numbers.

If $a_n \le b_n \le c_n$ holds for all $n > N$ (N is some index)

AND if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$ too.



EXAMPLE 4, p. 576 (a) $\dfrac{\cos(n)}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos(n)}{n} \le \dfrac{1}{n}$ and $-\dfrac{1}{n} \to 0$ and $\dfrac{1}{n} \to 0$.

(b) $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$ and $0 \to 0$ and $\dfrac{1}{n} \to 0$.

(c) $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)\dfrac{1}{n} \le \dfrac{1}{n}$ and $-\dfrac{1}{n} \to 0$ and $\dfrac{1}{n} \to 0$.

THM. (THE CTS. FN. THM) Let $\{a_n\} \subseteq \mathbb{R}$. Then if $a_n \to L$ and if f is a function that is cts. at L and defined at all a_n, then $f(a_n) \to f(L)$.

EXAMPLE.
5, p.577

From the last example, we know $\frac{n+1}{n} \to 1$. (We used the sum rule from Thm. 1). Let's use this to show $\sqrt{\frac{n+1}{n}} \to 1$. Observe: $f(x) := \sqrt{x}$ is continuous at $L := 1$, and is defined for all positive integers. So by the cts. fn. thm. for sequences, $f\left(\frac{n+1}{n}\right) \to f(1)$, i.e., $\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$.

EXAMPLE.
6, p. 577

Know $\frac{1}{n} \to 0$, and know $f(x) := 2^x$ is cts. at 0 and is defined on $\mathbb{R}$. Therefore, by the cts. fn. thm. for sequences, $f(\frac{1}{n}) \to f(0)$, i.e., $2^{1/n} \to 2^0 = 1$.

THM. Suppose $f(x)$ is defined for all $x \geq n_0$, and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then:

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

Proof. Assume the hypotheses of the theorem (i.e., $f(x)$ is defined for $x \geq n_0$, and $\{a_n\} \overset{\text{"is a subset of"}}{\subseteq} \mathbb{R}$ s.t. $\underbrace{a_n = f(n) \text{ for } n \geq \bigcirc\!\!\!\!\! n_0}_{(2)}$.

Suppose $\lim_{x\to\infty} f(x) = L$. This means, by the definition of the limit of a function, that:

$$\forall \varepsilon > 0, \ \exists \bigcirc\!\!\!\!\!\! M \in \mathbb{R} \text{ s.t. } x > M \Rightarrow |f(x) - L| < \varepsilon. \qquad (1)$$

Let $\varepsilon > 0$ be fixed, and let M be the real number as above. Let N be an integer greater than M and greater than n_0 (i.e., $N > \max\{M, n_0\}$). Then

$$n > N \underset{\substack{\uparrow \\ \text{as } N \geq M}}{\Rightarrow} n > M \underset{\substack{\uparrow \\ \text{by (1)}}}{\Rightarrow} |f(n) - L| < \varepsilon \underset{\substack{\uparrow \\ \text{as } N \geq n_0 \\ \text{and (2)}}}{\Rightarrow} \boxed{|a_n - L| < \varepsilon.}$$

So, the definition of the limit for sequences is satisfied.

EXAMPLE. 7, p. 577 Show that $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

Note that $f(x) := \frac{\ln(x)}{x}$ is defined for all $x \geq 1$ and agrees with the sequence $\frac{\ln(n)}{n}$ at positive integers. So by Thm. 4, if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} f(x)$. Now let's compute $\lim_{x\to\infty} f(x)$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln(x)}{x} \cancel{\underset{\text{Quot}}{\overset{?}{=}}} \frac{\lim_{x\to\infty} \ln(x)}{\lim_{x\to\infty} x} = \text{"}\frac{\infty}{\infty}\text{"}$$

✓ Indeterminate form "$\frac{\infty}{\infty}$".

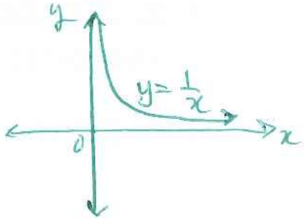
✓ $f(x) = \ln(x)$, $f'(x) = \boxed{1/x}$ } both exist everywhere except $x=0$
$g(x) = x$, $g'(x) = \boxed{1}$

✓ $g'(x) = 1 \neq 0$

$$\underset{\text{L'Hôp.}}{\overset{?}{=}} \lim_{x\to\infty} \frac{1/x}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x}$$

$$= 0.$$



We may therefore conclude that $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ too.

L2, ct'd.

EXAMPLE
8, p.578

Does $\{a_n\} = \left\{ \left(\frac{n+1}{n-1}\right)^n \right\}$ converge? If yes, to what?

Try directly taking the limit:

$$\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n \underset{\substack{\text{Substitut'n} \\ \text{(continuous} \\ \text{function)}}}{\overset{?}{=}} 1^\infty .$$

Indeterminate power, so use $\exp(\ln(\cdot))$:

$e^{\ln(\cdot)} = (\cdot)$

$$\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = \exp\left[\ln\left[\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n\right]\right]$$

$$= \exp\left[\lim_{n\to\infty}\left[\ln\left[\left(\frac{n+1}{n-1}\right)^n\right]\right]\right] \quad \text{as } \ln(\cdot) \text{ is cts.}$$

$$= \exp\left[\lim_{n\to\infty}\left(n \cdot \ln\left(\frac{n+1}{n-1}\right)\right)\right] \quad \text{has "}\infty \cdot 0\text{" form}$$

$$= \exp\left[\lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}\right] \quad \text{has "}\frac{0}{0}\text{" form}$$

$f(n) := \ln\left(\frac{n+1}{n-1}\right)$, $g(n) := \frac{1}{n}$

$f'(n) = \frac{(n-1)-(n+1)}{\cancel{(n-1)^2}}\left(\frac{\cancel{n-1}}{n+1}\right)$, $g'(n) = \frac{-1}{n^2}$

$= \frac{-2}{(n-1)(n+1)} = \frac{-2}{n^2+1}$

$$\overset{?}{\underset{\text{L'Hôp.}}{=}} \exp\left[\lim_{n\to\infty} \frac{-2/n^2+1}{-1/n^2}\right]$$

✓ Ind. form
✓ f', g' exist "everywhere"
✓ $g' \neq 0$

$$= \exp\left[\lim_{n\to\infty} \frac{2n^2}{n^2+1}\right]$$

$$= \exp\left[\lim_{n\to\infty} \left(\frac{2n^2}{n^2+1}\right)\underbrace{\left(\frac{1/n^2}{1/n^2}\right)}_{=1}\right]$$

$$= \exp\left[\lim_{n\to\infty} \frac{2}{1+\cancelto{0}{1/n^2}}\right]$$

$$= \exp[2] = e^2.$$

THM. 5 : COMMONLY OCCURRING LIMITS.

p. 578

① $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ ← Example 7

② $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Prove using Thm. 4

③ $\lim_{n\to\infty} x^{1/n} = 1 \quad (x>0)$

④ $\lim_{n\to\infty} x^{n} = 0 \quad (|x|<1)$

⑤ $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{n} = e^{x} \quad (\text{any } x)$

⑥ $\lim_{n\to\infty} \frac{x^{n}}{n!} = 0 \quad (\text{any } x)$

appendix of text for proofs.

Proof (3):

$$\lim_{n\to\infty} x^{1/n} = \exp\left[\ln\left[\lim_{n\to\infty} x^{1/n}\right]\right]$$

$$= \exp\left[\lim_{n\to\infty} \ln\left[x^{1/n}\right]\right]$$

$$= \exp\left[\lim_{n\to\infty} \frac{1}{n}\ln(x)\right]$$

$$= \exp\left[\ln(x) \lim_{n\to\infty} \frac{1}{n}\right] \leftarrow$$

$$= \exp\left[\ln(x)\cdot 0\right]$$

$$= \exp[0]$$

$$= 1.$$

- A sequence can also be defined RECURSIVELY, i.e.,
  - Give the initial term(s)
  - Give a rule (called a RECURSION FORMULA) for computing a term from those that precede it.

EXAMPLES. ①

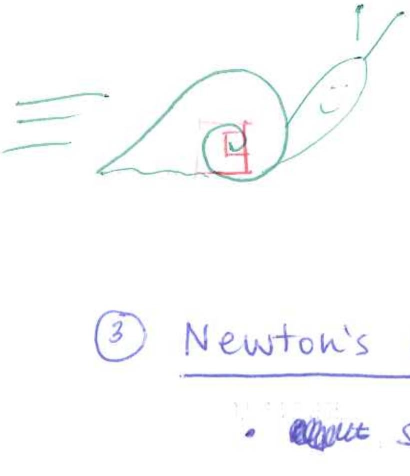
- $a_1 = 1$
- $a_n = n \cdot a_{n-1}$

$$a_1 = 1$$
$$a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2$$
$$a_3 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 6$$
$$a_4 = 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$
$$a_5 = 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$
$$\vdots$$
$$a_n = n!$$

② Fibonacci numbers

- $a_1 = 1, \ a_2 = 1$
- $a_{n+1} = a_n + a_{n-1}$

$$a_1 = 1$$
$$a_2 = 1$$
$$a_3 = a_2 + a_1 = 1 + 1 = 2$$
$$a_4 = a_3 + a_2 = 2 + 1 = 3$$
$$a_5 = a_4 + a_3 = 3 + 2 = 5$$
$$a_6 = a_5 + a_4 = 5 + 3 = 8$$
$$a_7 = a_6 + a_5 = 8 + 5 = 13$$
$$\vdots$$



$$a_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}, \quad \varphi \approx 1.618\ldots$$

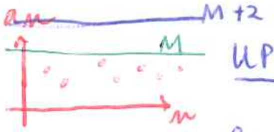
"Golden Ratio"

③ Newton's Method.

- start at x_0
- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

if we're lucky, these converge to a zero of $f(x)$.

DEF. A seq. $\{a_n\}$ is BOUNDED FROM ABOVE if $\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a_n \leq M$. In this case, M is called an UPPER BOUND for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the LEAST UPPER BOUND for $\{a_n\}$.



A seq. $\{a_n\}$ is BOUNDED FROM BELOW if there exists a number M such that for all n, $a_n \geq M$. Here, M is called a LOWER BOUND for $\{a_n\}$. If M is a lower bound for $\{a_n\}$ but no number greater than M is a lower bound for $\{a_n\}$, then M is the GREATEST LOWER BOUND for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is BOUNDED. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an UNBOUNDED SEQUENCE.

EXAMPLES.
11, p. 580

(a) $\{1, 2, 3, \ldots, n, \ldots\}$ has no upper bound.
But the greatest lower bound is $\underline{\;1\;}$.

(b) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$ has the least upper bound $\underline{\;1\;}$, and the greatest lower bound $\underline{\;1/2\;}$.

$\lim_{n\to\infty} \frac{n}{n+1}\left(\frac{1/n}{1/n}\right) = \lim_{n\to\infty} \frac{1}{1+\cancel{1/n}} = 1$ $\qquad \frac{n}{n+1} \overset{?}{\cancel{=}} 1 \iff n \cancel{=} n+1 \iff 0 \cancel{=} 1$

L2, CT'D.

Boundedness and convergence :

IF a sequence converges, $\Rightarrow$ THEN it is bounded. (1)

or the contrapositive: $[A \Rightarrow B]$ has contrapositive [NOT B $\Rightarrow$ NOT A]

IF a sequence is unbounded, THEN it diverges.

... but NOT the converse, i.e.,

NOT : ~~IF a sequence is bounded, THEN it converges.~~

An example: $\{1, -1, 1, -1, \dots, (-1)^{n-1}, \dots\}$

"Counterexample" diverges (cycles btwn. -1 and 1)

but : is bounded above by 1 (the LUB)

— " — below by -1 (the GLB)

Another counterex. : $a_n = \sin(n)$

Proof of (1): Suppose $a_n \to L$. Then by definition, taking $\varepsilon = 1$, $\underbrace{\exists}_{\text{there exists}} N \underbrace{\in}_{\text{"an element of"}} \overbrace{\mathbb{N}}^{\text{natural \#'s } \{1,2,3,\dots\}} \underbrace{\text{s.t.}}_{\text{such that}}\ n > N \Rightarrow |a_n - L| < 1$, i.e.,

"in the set" $L-1 < a_n < L+1$ whenever $n > N$.

Let $M > \max\{L+1, \underbrace{a_1, a_2, \dots a_N}_{\text{FINITELY MANY VALUES!}}\}$. Then by the definition of M, if $n \leq N$ then $M > a_n$, and as stated above, since $M > L+1$ and $L+1 > a_n$ for $n > N$, then we have also $n > N \Rightarrow M > a_n$. So $M > a_n$ for all $n \in \mathbb{N}$.

Similarly, let $m < \min\{L-1, a_1, \dots, a_N\}$. Then $m < a_n$ for $n \leq N$, and $m < L-1 < a_n$ for $n > N$, so $\forall n \in \mathbb{N}$, $m < a_n$. □

"for all"

DEF. A seq. $\{a_n\}$ is NONDECREASING if $a_n \leq a_{n+1}$ $\forall n$.

— " — NONINCREASING if $\forall n$, $a_n \geq a_{n+1}$.

— " — MONOTONIC if it is either nonincreasing or nondecreasing.

EXAMPLES. 12, p. 580

(a) $\{1, 2, \ldots, n, \ldots\}$ is nondecreasing

(b) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\}$ is also nondecreasing

(c) $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\}$ is nonincreasing

(d) $\{1, 1, 1, \ldots, 1, \ldots\}$ is both nondecreasing & nonincreasing

(e) $\{1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots\}$ is NOT MONOTONIC.

THEOREM: MONOTONIC SEQUENCE THM.

IF a seq. $\{a_n\}$ is BOTH bounded AND monotonic, THEN it converges.

NOTES: ① If nondecreasing, and bounded from above, then must have a least upper bound, and that is the limit.

② If nonincreasing and bd. from below, then must have a greatest lower bd., and that is the limit.

③ The converse is not true: $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, \frac{(-1)^{n+1}}{n}, \ldots\} \to 0$ but is not monotonic.

COUNTEREXAMPLE