-ecture 3: 10.2 (Infinite series), 10.3 (Integral Test)

10.2 : Infinite series.

An INFINITE SERIES is the sum of an infinite sequence of numbers:

$$\sum_{n=1}^{\infty} a_n := a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Cannot just keep adding infinitely many terms (+ is a binary operator) — so an infinite series is computed by taking the limit of the partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} \left[\sum_{n=1}^{k} a_n \right]$$

EXAMPLE: $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

1^{ST} partial sum: $\sum_{n=1}^{1} \frac{1}{2^{n-1}} = 1$

2^{ND} ——"——: $\sum_{n=1}^{2} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} = \frac{3}{2}$

3^{RD} ——"——: $\sum_{n=1}^{3} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

4^{TH} partial sum: $\displaystyle\sum_{n=1}^{4} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

⋮

n^{th} partial sum: $\displaystyle\sum_{n=1}^{n} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$

$$= \frac{2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 1}{2^{n-1}}$$

$$= \boxed{\frac{2^n - 1}{2^{n-1}}}$$

⋮

limit of partial sums:

$$\left\{ 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots, \frac{2^n-1}{2^{n-1}}, \ldots \right\}$$ ← seq. of partial sums

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = \lim_{n\to\infty} 2 - \underbrace{\frac{1}{2^{n-1}}}_{0} = 2.$$

So, $\boxed{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots = 2.}$

Note: Think of the $n^{\underline{th}}$ partial sum as the n^{th} approximation of the series value, if it converges.

, ct'd.

DEF. (p. 585) Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is called an INFINITE SERIES. The number a_n is the n^{TH} TERM of the series.

The sequence $\{s_k\}$ whose terms are defined by:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_k = a_1 + \dots + a_k = \sum_{n=1}^{k} a_n$$

$$\vdots$$

is the sequence of PARTIAL SUMS of the series, the number S_k being the k^{TH} PARTIAL SUM. If the seq. of partial sums converges to a limit L, we say that the series CONVERGES and that its SUM is L. We write:

$$\sum_{n=1}^{\infty} a_n := a_1 + a_2 + \dots + a_n + \dots = L.$$

If the seq. of partial sums does not converge, then we say that the series DIVERGES.

Geometric series.

$$\sum_{n=0}^{\infty} a r^n = a + ar + ar^2 + \cdots + ar^n + \cdots$$

some real #

some rational number.

$$= a(1 + r + r^2 + r^3 + \cdots + r^n + \cdots)$$

$$= \frac{a}{1-r}, \quad |r| < 1$$

$$\sum_{n=1}^{\infty} a r^{n-1}$$

If $|r| \geq 1$, the series diverges.

EXAMPLE: $1 + \frac{1}{2} + \frac{1}{4} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

and here, $a = 1$

$r = 1/2$, so $|r| < 1$ and

the series converges:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\overset{a}{1}}{1 - \underbrace{\tfrac{1}{2}}_{r}} = \frac{1}{\frac{1}{2}} = 2.$$

EXAMPLE. $10 - \frac{10}{3} + \frac{10}{9} - \frac{10}{27} + \cdots + 10\left(\frac{-1}{3}\right)^n + \cdots = \sum_{n=0}^{\infty} 10\left(\frac{-1}{3}\right)^n$

Here, $a = 10$

$r = -1/3$, $|r| < 1$, so series converges

So $\sum_{n=0}^{\infty} 10\left(\frac{-1}{3}\right)^n = \frac{10}{1+\frac{1}{3}} = \frac{10}{\frac{4}{3}} = \frac{30}{4} = \boxed{\frac{15}{2}}$

Proof for convergence of geom. series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

The n^{th} partial sum: $S_n = a + ar + ar^2 + \dots + ar^n$.

$$S_n = a + ar + \dots + ar^n = \sum_{k=1}^{n} ar^k$$

$$r \cdot S_n = ra + ar^2 + \dots + ar^{n+1}$$

So
$$\underbrace{S_n - r \cdot S_n}_{} = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^n}$$
$$- \cancel{ar} - \cancel{ar^2} - \dots - \cancel{ar^n} - ar^{n+1}$$
$$= \underbrace{a - ar^{n+1}}_{}$$

So
$$S_n(1-r) = a(1-r^{n+1})$$

$$S_n = \frac{a(1-r^{n+1})}{1-r}$$

and if $r \neq 1$, then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{a}{1-r}\right)(1-r^n) = \frac{a}{1-r} \lim_{n\to\infty} (1-r^n).$$

Now, if $|r|<1$ then $\lim_{n\to\infty} r^n = 0$; else, $r^n \to \infty$.

So if $r<1$, $\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} S_n = \frac{a}{1-r}$.

 Now, if $|r| = 1$:

Case I
$r=1$

$$\sum_{n=0}^{\infty} a(1)^n = a + a(1) + a(1)^2 + \dots$$

and the n^{th} partial sum: $S_n = a + a(1) + \dots + a(n)$
$= a(n)$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} a(n) = \infty.$

Therefore, series does not converge.

Case II.
$r=-1$

$$\sum_{n=0}^{\infty} a(-1)^n = a - a + a - a + \dots$$

and the n^{th} p.s.: $S_n = \overbrace{a - a + a - \dots}^{n \text{ many times}}$

$$= \begin{cases} 0, & n - \text{even} \\ a, & n - \text{odd} \end{cases}$$

So $\lim_{n\to\infty} S_n$ also does not exist, since the partial sums alternate between 0 and a.

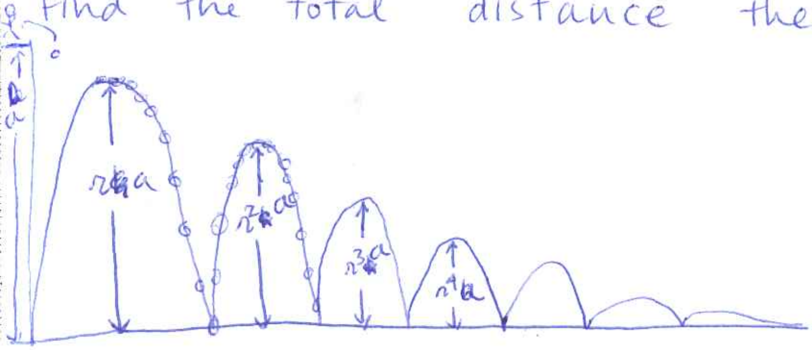
Therefore, series does not converge.

In sum:

$$\boxed{\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r}, & |r|<1 \\ (\text{divergent}), & |r| \geq 1. \end{cases}}$$

EXAMPLE. 3, p. 587

You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down.



$$d = h + 2rh + 2r^2h + \dots$$

$$= h + \sum_{n=1}^{\infty} 2hr^n$$

MUST SUBTRACT OFF 0^{TH} TERM!

$$= h + \left[\sum_{n=0}^{\infty} 2hr^n\right] - 2h$$

$$= \left[\sum_{n=0}^{\infty} 2hr^n\right] - h$$

$$= \frac{2h}{1-r} - h$$

$$= \frac{2h - h(1-r)}{1-r}$$

$$= \frac{h(1+r)}{1-r}$$

Note: not physically realistic, but a good approx.

e.g., $\left.\begin{array}{l} h = 6m \\ r = \frac{2}{3} \end{array}\right\} \Rightarrow d = 6m\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = 6m\left(\frac{\frac{5}{3}}{\frac{1}{3}}\right) = 6m(5) = 30\,m.$

EXAMPLE. Express the repeating decimal $5.23\overline{23}\ldots$ as the ratio of two integers.

4, p. 587

$$5.\underbrace{23}\underbrace{23}\ldots = 5 + \frac{23}{100} + \frac{23}{10{,}000} + \frac{23}{1{,}000{,}000} + \cdots$$

$$= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots \frac{23}{100^n} + \cdots$$

$$= 5 + \frac{23}{100}\left(1 + \frac{1}{(100)^2} + \frac{1}{(100)^3} + \cdots\right)$$

$$= 5 + \frac{23}{100}\left[\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n\right] \qquad a = 1, \quad r = 1/100$$

$$= 5 + \frac{23}{100}\left[\frac{1}{1 - \frac{1}{100}}\right]$$

$$= 5 + \frac{23}{\cancel{100}} \cdot \frac{\cancel{100}}{99}$$

$$= 5 + \frac{23}{99} = \boxed{\frac{518}{99}}$$

Telescoping series.

EXAMPLE. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

[see: $\frac{1}{n}-\frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$] ✓

$$= \underbrace{\left(1-\tfrac{1}{2}\right)}_{n=1}+\underbrace{\left(\tfrac{1}{2}-\tfrac{1}{3}\right)}_{n=2}+\underbrace{\left(\tfrac{1}{3}-\tfrac{1}{4}\right)}_{n=3}+\cdots$$

So, $$S_n = \left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\cancel{\ }\cdots\cancel{+}\left(\cancel{\tfrac{1}{n-2}}-\cancel{\tfrac{1}{n-1}}\right)+\left(\cancel{\tfrac{1}{n-1}}-\cancel{\tfrac{1}{n}}\right)+\left(\cancel{\tfrac{1}{n}}-\frac{1}{n+1}\right)$$

$$= 1-\frac{1}{n+1} = n^{th} \text{ partial sum}$$

and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1-\underbrace{\frac{1}{n+1}}_{\to 0} = 1$, so $\boxed{\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1}$

n^{th} term test FOR DIVERGENT SERIES.

THEOREM. IF $\sum_{n=1}^{\infty} a_n$ converges, THEN $a_n \to 0$.

Why? Suppose $\sum_{n=1}^{\infty} a_n = S$. Then ~~[illegible]~~ $\boxed{a_n = S_n - S_{n-1}}$ ← partial sums

$S_n \to S$, $S_{n-1}\to S$, so $a_n \to S-S=0$.

CONTRAP. OF $A\Rightarrow B$ and is ~~[illegible]~~ $\neg B \Rightarrow \neg A$; CONVERSE IS $\neg A \Rightarrow \neg B$.

CONTRAPOSITIVE: IF $a_n \not\to 0$, THEN $\sum_{n=1}^{\infty} a_n$ DIV.

NOTE: ~~[illegible]~~ The CONVERSE is NOT true! ~~IF $a_n \to 0$ THEN $\sum_{n=1}^{\infty} a_n$ converges~~

COUNTEREXAMPLE: $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ DOES NOT CONVERGE

HARMONIC SERIES DIV.

EXAMPLE: 7, p. 588

(a) $n^2 \to \infty$, so $\sum_{n=1}^{\infty} n^2$ diverges

(b) $\frac{n+1}{n} \to 1^{\neq 0}$, so $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges

(c) $\lim_{n\to\infty} (-1)^{n+1}$ does not exist, so $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges

(d) $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$, so $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges.

Combining series.

Can add/subtract convergent series termwise, or constant multiples:

THM. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

1. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
2. $\sum_{n=1}^{\infty} k a_n = kA$, any constant k
3. $\sum_{n=1}^{\infty} a_n - b_n = \sum_{n=1}^{\infty} a_n + (-1) b_n = A - B$

Proofs are friendly! (HW)

COR. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} k a_n$ diverges.

COR. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges,

then $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n - b_n)$ both also diverge.

BUT !! If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both DIVERGE, we don't know anything abt. the (possible) convergence of $\sum_{n=1}^{\infty} a_n + b_n$ or $\sum_{n=1}^{\infty} a_n - b_n$:

EX 1

$a_n = 1$

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + \dots$$

$$\sum_{n=1}^{\infty} (-1) = -1 - 1 - 1 \dots$$

$b_n = -1$

both diverge, but $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} 1 + (-1)$

$$= \sum_{n=1}^{\infty} 0 = 0$$

(does converge)

EX 2

$a_n = 1$

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + \dots$$

$$\sum_{n=1}^{\infty} 2 = 2 + 2 + \dots$$

$b_n = 2$

both diverge, and $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} 1 + 2 = \sum_{n=1}^{\infty} 3$

(diverges)

- Adding or deleting finitely many terms does not change whether (or not) the series converges (but will usually change the sum).

e.g.: $$\sum_{n=1}^{\infty} \frac{1}{5^n} = \underbrace{\frac{1}{5} + \frac{1}{25} + \frac{1}{125}} + \overbrace{\sum_{n=4}^{\infty} \frac{1}{5^n}}^{\text{"tail" of series}}.$$

So $$\sum_{n=4}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{1}{5^n} \underbrace{- \frac{1}{5} - \frac{1}{25} - \frac{1}{125}}.$$

- Can reindex without altering convergence:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + \cdots \quad \text{RAISES START INDEX}$$

$$= \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + \cdots \quad \text{LOWERS START INDEX}$$

EXAMPLE: Geometric series.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

$$= \sum_{n=(1-1)}^{\infty} ar^{(n+1)-1} = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

$$= \sum_{n=(5+1)}^{\infty} ar^{(n-5)-1} = \sum_{n=6}^{\infty} ar^{n-6} = a + ar + ar^2 + \cdots$$

10.3 : The Integral Test.

- Nondecreasing partial sums.

Suppose $\sum_{n=1}^{\infty} a_n$ has $a_n \geq 0$ for all n.

Then as usual, $S_{n+1} = S_n + a_n \geq S_n$, so

$$S_1 \leq S_2 \leq S_3 \leq \cdots \leq S_n \leq S_{n+1} \leq \cdots$$

So we have a NONDECREASING SEQUENCE.

Recall the MONOTONIC SEQUENCE THM:

IF $\{a_n\}$ is monotonic and bounded,
THEN a_n converges.

This means what to us as far as series are concerned?

or for all but finitely many?

COR. IF $\sum_{n=1}^{\infty} a_n$ has $a_n \geq 0$ $\forall n$, AND $\{S_n\}$ is bd. from above,
THEN $\sum_{n=1}^{\infty} a_n$ converges!

CONVERSE: IF $\sum_{n=1}^{\infty} a_n$ has $a_n \geq 0$ $\forall n$, AND $\sum_{n=1}^{\infty} a_n$ converges,
THEN $\{S_n\}$ is bd. from above.

True? YES

- When an implication and its converse are both true, we call the statement an equivalency, or an "if and only if", abbreviated "iff"; or $\Leftrightarrow$

> A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges
>
> IF AND ONLY IF
>
> its partial sums are bounded from above.

Aside: Summary of the logic we know thus far:

Implication/if-then statement: IF A, THEN B.

$A \Rightarrow B$ "A implies B"

... is logically equivalent to the contrapositive: IF NOT B, THEN NOT A

$\neg B \Rightarrow \neg A$

example: If I go to WPI, THEN my mascot is a goat.

... is logically equivalent to:

If my mascot is NOT a goat, THEN I do NOT go to WPI.

... but we don't know whether the converse holds: $B \Rightarrow A$

(?) IF my mascot is a goat, THEN I go to WPI (?)

A counterexample to the converse is any school that has a goat as a mascot but is not WPI.

(i.e., to disprove $B \Rightarrow A$, show something where B is true, A is not)

... but if no counterexample exists, the converse is true, and we have an if-and-only-if statement: $A \Leftrightarrow B$

I go to WPI IF AND ONLY IF my mascot is a goat.

The HARMONIC SERIES $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ DIVERGES.

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = 1/2} + (\cdots)$$

The 1st two terms sum to $\frac{3}{2}$.

The next two terms' sum is greater than $\frac{1}{2}$

— " — four — " — $\frac{1}{2}$

— " — eight — " — $\frac{1}{2}$

⋮

The next 2^n terms' sum is greater than $\frac{1}{2}$.

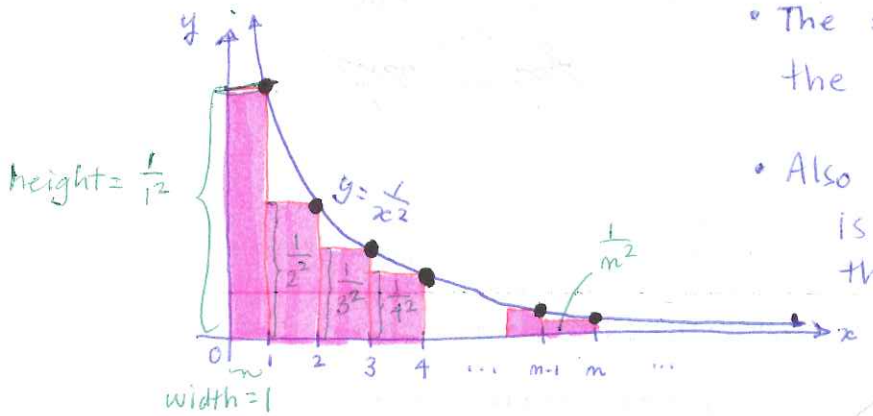
So the $(2^n)^{th}$ partial sum is greater than $\frac{3+n}{2}$, which is NOT bounded from above.

So $\boxed{\text{the harmonic series diverges.}}$

Note that $\frac{1}{n} \to 0$, though ...

The Integral Test.

EXAMPLE 2, p. 594 Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?



- The sum of the pink areas is the sum of the series.
- Also, the sum of the pink areas is less than the area under the curve! (The integral)

So,
$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx + \underbrace{f(1)}_{=1}$$
$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx + 1$$
$$= \lim_{b\to\infty} \left[-\frac{1}{x} \Big|_1^b \right] + 1$$
$$= \lim_{b\to\infty} \left[-\underbrace{\frac{1}{b}}_{0} + 1 \right] + 1 = 1 + 1 = 2.$$

So the sum is bounded above by 2, and the series converges.

NOTE!! $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\boxed{\approx 1.645...}} \neq \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}_{=1}$

INTEGRAL TEST. Suppose $\{a_n\}$ is a seq. of positive terms, and $a_n = f(n)$, with f - cts. (CONTINUOUS), positive, decreasing for all $x > N \in \mathbb{N}$ (an elt. of the nat. #'s). Then

$$\sum_{n=N}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,dx \quad \text{either both converge, or both diverge.}$$

(see proof in text?)

EXAMPLE: P-SERIES ("POWER SERIES")

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots \quad \begin{cases} \text{conv.}, & p > 1 \\ \text{div.}, & p \le 1 \end{cases}$$

TO WHAT?

$p=1$: HARMONIC SERIES

proof: $f(x) = \dfrac{1}{x^p}$ is cts, positive, decreasing (if $p > 0$).

So
$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{1}{1-p} \left(x^{1-p}\right)\Big|_1^b \right]$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right) = \infty \text{ if } 0 < p \le 1$$

$$= -\frac{1}{1-p} = \frac{1}{p-1}, \text{ if } p > 1.$$

If $p \le 0$, then n^{th} term: $\displaystyle \lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{|p|} = \infty$.

NOTE: This does NOT say $\displaystyle \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{p-1}$. (!!)

- The p-series with $p=1$ is the harmonic series, which DIVERGES.
  - But take $p = 1.00000001$, and the p-series converges.
  - Partial sums for harmonic series grow, but grow SLOWLY: it takes 178 million terms for S_n to exceed 20.

EXAMPLE OF INTEGRAL TEST.

5, p. 596

(a) $$\sum_{n=1}^{\infty} ne^{-n^2} < \int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{e^{x^2}}\,dx$$

let $u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$u(1) = 1$

$u(b) = b^2$

$$= \lim_{b\to\infty} \left[\int_1^{b^2} \frac{1}{2e^u}\,du \right]$$

$$= \lim_{b\to\infty} \left[-\frac{1}{2e^u} \Big|_1^{b^2} \right]$$

$$= \lim_{b\to\infty} \left(\underbrace{-\frac{1}{2e^{b^2}}}_{\to 0} + \frac{1}{2e} \right) = \frac{1}{2e}.$$

So, the series converges. BUT WE DON'T KNOW WHAT TO!

(b) $$\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}} < \int_1^{\infty} \frac{1}{2^{\ln(x)}}\,dx = \lim_{b\to\infty} \int_1^b 2^{-\ln(x)}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$x = dx = e^u\,du$

$u(1) = 0$

$u(b) = \ln(b)$

$$= \lim_{b\to\infty} \int_0^b \frac{e^u}{2^u}\,du$$

$$\Big) = \left(\frac{e}{2}\right)^u$$

$$= \lim_{b\to\infty} \left[\frac{1}{\ln\left(\frac{e}{2}\right)} \left[\underbrace{\left(\frac{e}{2}\right)^{\ln(b)}}_{\to\infty} - 1 \right] \right]$$

$e \approx 2.71\ldots$

$$= \infty.$$

So the series diverges.

Error Estimation.

Sometimes we know a series converges, but we don't know what to. Can estimate it using partial sums:

$$\sum_{n=1}^{\infty} a_n = S \quad \text{and partial sums are } \{s_n\}.$$

Define the REMAINDER $\quad R_n := S - s_n$

$$= a_{n+1} + a_{n+2} + \dots$$

Then $R_n = a_{n+1} + a_{n+2} + \dots \geq \int_{n+1}^{\infty} f(x)\,dx$, with $f(n) = a_n \;\forall n$. ← lower bound

and $R_n = a_{n+1} + a_{n+2} + \dots \leq \int_{n}^{\infty} f(x)\,dx$ ← upper bound

EXAMPLE
8, p.597

Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using $n=10$.

Compute $\int_k^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_k^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x} \Big|_k^b \right]$

$$= \lim_{b\to\infty} \left[\underbrace{-\frac{1}{b}}_{\to 0} + \frac{1}{k} \right] = \frac{1}{k}.$$

So $\frac{1}{11} \leq R_{10} \leq \frac{1}{10}$, and since $R_{10} = S - s_{10}$,

$$s_{10} + \frac{1}{11} \leq S \leq s_{10} + \frac{1}{10}.$$

Now, $s_{10} = 1 + \frac{1}{4} + \dots + \underset{1/10^2}{\frac{1}{100}} \approx 1.54977$, so $1.64068 \leq S \leq 1.64977$

and $S \approx \frac{1}{2}(1.64068 + 1.64977) = 1.6452.$ [Really, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493$]