Lecture 4: Infinite Series and convergence tests. (10.2-10.5)

Recap: infinite series & convergence tests.

- For an infinite series $\sum_{n=1}^{\infty} a_n$, the PARTIAL SUMS are defined:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_N = \sum_{n=1}^{N} a_n \longleftarrow \text{the "}N^{th}\text{ partial sum"}$$
$$\vdots$$

or $S_{N+1} = S_N + a_{N+1}$

and the series converges if and only if its seq. of partial sums converges, and its sum is that limit
the limit of the partial sum

EXAMPLE: ① $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 \cdots$ has $S_1 = (-1)^1 = -1$

$$S_2 = (-1)^1 + (-1)^2 = -1 + 1 = 0$$
$$S_3 = (-1)^1 + (-1)^2 + (-1)^3 = -1 + 1 - 1 = -1$$
$$\vdots$$
$$S_N = \begin{cases} -1, & N\text{-odd} \\ 0, & N\text{-even} \end{cases}$$

and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$ does not exist

EXAMPLES (2) $2 + \frac{2}{3} + \frac{2}{9} + \dots + \frac{2}{3^{n-1}} + \dots = \sum_{n=1}^{\infty} \frac{2}{3^{n-1}}$

has $S_1 = 2$

$S_2 = 2 + \frac{2}{3} = \frac{2(3)+2}{3} = \frac{8}{3}$

$S_3 = 2 + \frac{2}{3} + \frac{2}{9} = 2\,\frac{9+3+1}{9} = 2\,\frac{13}{9} = \frac{26}{9}$

$S_4 = 2\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right) = 2\,\frac{27+9+3+1}{27} = 2\,\frac{40}{27} = \frac{80}{27}$

$S_5 = 2\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}\right) = 2\,\frac{81+27+9+3+1}{81} = 2\,\frac{121}{81} = \frac{242}{81}$

$\vdots$

$S_N = \frac{3^N - 1}{3^{N-1}}$

So $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{3^N - 1}{3^{N-1}} = \lim_{n\to\infty}\left[3 - \frac{1}{3^{N-1}}\right] = 3$

So the partial sums converge to 3, and $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$.

(converges to 3 as well)

- **Geometric series**

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \sum_{n=0}^{\infty} a \cdot r^{n} \quad \text{converges to } \frac{a}{1-r} \text{ if } |r|<1$$

$$\text{diverges if } |r| \geq 1$$

- **p-series.**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p>1$$

diverges if $p \leq 1$. (HARMONIC SERIES DIVERGES)

- **Telescoping series.**

where partial sums have all but 1^{st} and last "terms" cancel:

EXAMPLE. $\sum_{n=1}^{\infty}\left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$ has $S_k = \left(\frac{3}{1^2} - \cancel{\frac{3}{2^2}}\right) + \left(\cancel{\frac{3}{2^2}} - \cancel{\frac{3}{3^2}}\right) \cancel{+} \cancel{\cdots} \left(\cancel{\frac{3}{(k-1)^2}} - \frac{3}{k^2}\right)$

$= \sum_{n=1}^{\infty}\left(\frac{6n+3}{n^2(n+1)^2}\right)$ $\qquad = 3 - \frac{3}{k^2}$

and $\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(3 - \frac{3}{k^2}\right) = 3$,

0

so the series converges to 3.

- **Combining series.**

  - If $\sum a_n = A$ and $\sum b_n = b$ both converge:

    - $\sum(a_n + b_n) = (\sum a_n) + (\sum b_n) = A + B$ sum rule

    - $\sum(k a_n) = k(\sum a_n) = kA$ product rule for constants

~~$(1+2+3+4)(-1-2-3-4) \neq (-1)(1) + (-2)(2) + \cdots$~~

- Every nonzero constant multiple of a divergent series diverges too.

- If Σa_n converges and Σb_n diverges, then $\Sigma(a_n+b_n)$ and $\Sigma(a_n-b_n)$ both diverge.

- If Σa_n and Σb_n both diverge, can't say anything about either $\Sigma(a_n+b_n)$ or $\Sigma(a_n-b_n)$.

- Adding or deleting finitely many terms doesn't affect whether a series converges – just (possibly) its sum

- Can re-index: $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h}$ to raise start index

  $\displaystyle = \sum_{n=1-h}^{\infty} a_{n+h}$ to lower start index.

- n^{th} term test for divergence:

  If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

  $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$ but $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

- Integral test:

  If $a_n = f(n)$ where f is
  - positive
  - continuous
  - decreasing

  for all $x, n \geq N$

  Then $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

  diverge/
  "converge together"

Now, some more convergence tests:

THEOREM: COMPARISON TEST

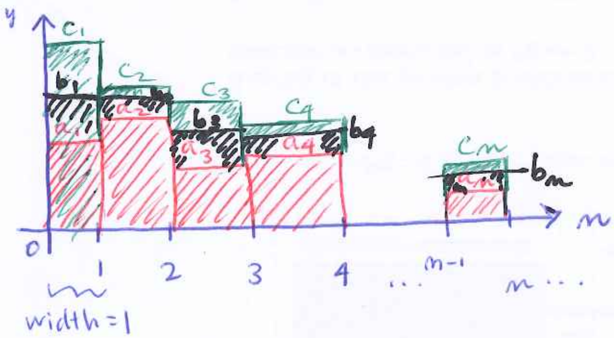
Suppose a_n, b_n, and c_n are all non-negative (for all $n > N$) and for all $n > N$,

$$a_n \le b_n \le c_n$$

Then both of the following implications are true:

If $\sum c_n$ converges, $\sum b_n$ converges.

If $\sum a_n$ diverges, $\sum b_n$ diverges



Key idea: sum of series is the same as the sum of the areas of the rectangles.

If $\sum c_n$ is finite, that forces $\sum b_n$ to be finite (and $\sum a_n$ as well), so $\sum b_n$ converges (and $\sum a_n$)

If $\sum a_n$ is infinite, that forces $\sum b_n$ (and $\sum c_n$) to be infinite too, so $\sum b_n$ (and $\sum c_n$) diverges.

NOTE: Useless if $\sum a_n$ converges or $\sum c_n$ diverges

EXAMPLES. (a) $\sum_{n=1}^{\infty} \frac{5}{5n-1} = \sum_{n=1}^{\infty} \frac{5}{5n-1} \overbrace{\left(\frac{1/5}{1/5}\right)}^{=1} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n-1/5}}_{a_n}$

and $n - \frac{1}{5} < n$, so $\frac{1}{n-\frac{1}{5}} > \frac{1}{n}$. Because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\frac{5}{5n-1} > \frac{1}{n}$, we know by the comparison test that $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges too.

(b) $\underbrace{\cancel{5 + \frac{2}{3} + \frac{1}{7}}}_{\text{TRUNCATE THESE}} + \underbrace{1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{3}} + \frac{1}{16+\sqrt{4}} + \cdots + \frac{1}{2^n+\sqrt{n}} + \cdots}$

FOR THESE, $\frac{1}{2^n+\sqrt{n}} < \frac{1}{2^n}$,

and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2} < 1$, so it converges; thus, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$ converges too, and so does our original (un-truncated) ~~sequence~~ series.

THM: LIMIT COMPARISON TEST.

Suppose $a_n > 0$ and $b_n > 0$ for all $n \geq N$.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then Σa_n and Σb_n either both conv. or both div.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and Σb_n conv., then Σa_n conv. too

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and Σb_n div., then Σa_n diverges too

NOTES: In case (2), if Σb_n div, then test is INCONCLUSIVE.

— " — (3), if Σb_n conv., ——— " ———.

EXAMPLE. (a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$. Guess: $\left(\frac{2n+1}{n^2+2n+1}\right)\left(\frac{1/n}{1/n}\right) = \frac{2+\cancel{1/n}}{n+2+\cancel{1/n}}$

$\to \frac{1}{n}$, as large terms dominate.

so try $b_n := \frac{1}{n}$, and note $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so trying case (1) or (3).

Then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+1}{n^2+2n+1}\right)}{(1/n)}$$

$$= \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1}\left(\frac{1/n^2}{1/n^2}\right) = \lim_{n\to\infty} \frac{2+\cancel{1/n}^{\to 0}}{1+\cancel{2/n}^{\to 0}+\cancel{1/n^2}^{\to 0}} = 2$$

and we have case (1), so $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ diverges.

So, we have two more tests:

- n^{th} term for divergence
- Integral test

NEW
- Comparison test
- Limit comparison test.

And we know convergence/divergence for:

- Geometric series $\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r}$, $|r|<1$
  DIV., else
- p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv. if $p>1$
  div. else
- Telescoping series

... And we can always use the definition of convergence:

$$\sum_{n=1}^{\infty} a_n = A \iff \lim_{N\to\infty} \underbrace{\left[\sum_{n=1}^{N} a_n \right]}_{\text{"partial sum"}} = A$$