Lecture 6: Taylor & Binomial Series (10.9, 10.10) & Review. 1

ANNOUNCEMENTS:

- Homework 2 due tomorrow (Friday), 11:59 p.m.
- Homework 3 posted, but DON'T TURN IN — sol'ns available Sat/Sunday
- There will be a MIDTERM EXAM ON TUESDAY next week! (We will review during the 2nd half tonight)

10.9: Convergence of Taylor series.

Recall: The TAYLOR SERIES generated by $f(x)$ about $x=a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

THEOREM (TAYLOR'S FORMULA)

If $f(x)$ has derivatives of all orders in some open nbd. (neighborhood) I of $x=a$,

Then for each positive integer n and for each $x \in I$,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] + \boxed{R_n(x)},$$ "error term" "remainder term"

where for some c between a and x, $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

- Proof on p. 636 — not on exam, but interesting (?)
- Error $R_n(x)$ has the form of the other terms, except the derivative is evaluated at c.
- c depends on x and on a.

DEF. If $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in I$,

Then we say that the Taylor series CONVERGES to f on I, and we write:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

EXAMPLE. 1, p. 632 Show that the Taylor series generated by $f(x) = e^x$ at $x=0$ converges to $f(x)$ for every real value of x.

Check: ✓ $f(x)$ is infinitely diff'ble on $I = (-\infty, \infty)$?

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$$

$$f''(x) = e^x$$

$$\vdots$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$$

$$f^{(n+1)}(x) = e^x \Rightarrow f^{(n+1)}(c) = e^c$$

So, taking $a=0$, we write Taylor's formula:

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$= \left[\sum_{k=0}^{n} \frac{x^k}{k!}\right] + \boxed{\frac{e^c x^{n+1}}{(n+1)!}}.$$ REMAINDER TERM

Now, $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{e^c x^{n+1}}{(n+1)!} = e^c \underbrace{\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!}}_{\text{THEOREM 10.1.5}} = e^c \cdot 0 = 0.$

THEOREM 10.1.5 : $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

So for $x \in (-\infty, \infty)$, $\boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$.

L6, ct'd.

EXAMPLE
1, p. 632, ct'd.

$= 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots$ 3

Cool result: $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, and if we truncate this series at n many terms, we use the Taylor remainder term to know the error:

$$e = \sum_{k=0}^{n} \frac{1}{k!} + R_n(1)$$

$$= \sum_{k=0}^{n} \frac{1}{k!} + \boxed{\frac{e^c}{(n+1)!}}, \text{ where } c \in (0,1).$$

If $c \in (0,1)$, then $e^c \in (1, e)$, so the error when truncating at n terms is bounded above:

$$R_n(1) = \frac{e^c}{(n+1)!} < \frac{e}{(n+1)!}$$

Gives an easy way to know how many terms are needed for a certain precision in the approximation.

- Taylor series convergence is important — a Taylor series is, after all, just a POWER SERIES, and we have (had...) theorems:
  - Termwise differentiation + integration on the interval of convergence
  - Addition, subtraction, and *multiplication* of series on the intersec'n of their int. of convergence

EXAMPLE 4, p. 634

Using known series, write the 1st few terms of the Taylor series for: (a) $\frac{1}{3}(2x + x\cos(x))$ (b) $e^{x}\cos(x)$.

Known: $$e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

(a) $$\frac{1}{3}(2x + x\cos(x)) = \frac{2}{3}x + \frac{1}{3}x\left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\right]$$

$$= \frac{2}{3}x + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{3(2k)!}$$

$$= \frac{2}{3}x + \left[\frac{x}{3} - \frac{x^3}{3\cdot 2!} + \frac{x^5}{3\cdot 4!} - \frac{x^7}{3\cdot 6!} + \dots\right]$$

$$= \boxed{x - \frac{x^3}{6} + \frac{x^5}{3\cdot 4!} - \frac{x^7}{3\cdot 6!} + \dots}$$

EXAMPLE 4, p. 634 ct'd.

(b) $e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right)$

$\cos(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

because the Taylor series for e^x and $\cos(x)$ both converge $\forall x$

so

$$e^x \cos(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) +$$

$$+ \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots\right) +$$

$$+ \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{2 \cdot 4!} - \frac{x^8}{2 \cdot 6!} + \cdots\right) + \left(\frac{x^3}{6} - \frac{x^5}{12} + \frac{x^7}{6 \cdot 4!} - \frac{x^9}{6 \cdot 6!} + \cdots\right)$$

$$+ \cdots$$

$$= 1 + x + x^2\cancel{\left(-\frac{1}{2} + \frac{1}{2}\right)} + x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + x^4\left(\frac{1}{4!} - \frac{1}{4} + \frac{1}{4!}\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

(or, use the def'n in multiplicat'n th'm — this will give an equivalent result, provided correct arithmetic)

10.10: Binomial series + Applicat'ns of Taylor series.

Let's find the Taylor series for $f(x) = (1+x)^m$, abt. $a=0$:

$$f(x) = (1+x)^m \overset{\text{implies}}{\Rightarrow} f(0) = (1+0)^m = 1$$

$$f'(x) = m(1+x)^{m-1} \Rightarrow f'(0) = m(1+0)^{m-1} = m$$

$$f''(x) = m(m-1)(1+x)^{m-2} \Rightarrow f''(0) = m(m-1)$$

$$f'''(0) = m(m-1)(m-2)$$

$$\vdots$$

$$f^{(k)}(0) = m(m-1)(m-2)\cdots(m-k+1)$$

= the set of "natural #'s" $\{1,2,3,\dots\}$

If $m \in \mathbb{N}$, then after $(m+1)$ many ~~terms~~ derivs, one of the coefficients of $f^{(k)}(0)$ is $(m-(m+1)+1) = 0$, so the series is finite.

"the set of real #'s, except the natural #'s"

If $m \in \mathbb{R} \setminus \mathbb{N}$, then it is an infinite series that does converge to $f(x)$ (can show elsewhere), for $|x| < 1$.

So...

> BINOMIAL SERIES for $x \in (-1,1)$, $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, where...
>
> "binomial coefficient"
> "m-choose-k"
>
> - $\binom{m}{0} = 1$, • $\binom{m}{1} = m$
> - $\binom{m}{2} = \frac{m(m-1)}{2!}$, $\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$

EXAMPLE 1, p. 639 Take $m=-1$. Then $f(x) = (1+x)^{-1} = \dfrac{1}{1+x}$.

We compute coefficients:

$$\binom{-1}{0} = 1, \quad \binom{-1}{1} = -1, \quad \binom{-1}{2} = \frac{-1(-2)}{2!} = 1,$$

$$\binom{-1}{3} = \frac{-1(-2)(-3)}{3!} = -1, \quad \binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = (-1)^k \frac{\cancel{k!}}{\cancel{k!}}$$

So, $$f(x) = \frac{1}{1+x} = \sum_{k=0}^{\infty} \binom{-1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

$$= \boxed{1 - x + x^2 - x^3 + x^4 - \ldots}$$

Uses of Taylor series.

- Nonelementary integrals

EXAMPLE 3, p. 640 Express $\int \sin(x^2)\,dx$ as a power series.

Well, $$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(the convergence of this Taylor series for all $x \in \mathbb{R}$ was proven in Example 2, p. 633 — on HW, you need to show!),

so $$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

by the substitution thm. for convergent power series on pg. 622, since $f(x) := x^2$ is cts. everywhere

continuous

$\boxed{\dfrac{\text{EXAMPLE}}{3,\ \text{p. } 640}\ \text{ct'd.}}$

$$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

so,

$$\int \sin(x^2)\,dx = \int \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}\right] dx = \int \left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right] dx$$

$$= \sum_{k=0}^{\infty} \left[\int \frac{(-1)^k x^{4k+2}}{(2k+1)!}\,dx\right] = \left[\int x^2\,dx\right] + \left[\int -\frac{x^6}{3!}\,dx\right] + \cdots$$

This is a big step! We can only switch the order of summation & integration because our series is a convergent power series – see Theorem 22, p. 623 "Termwise integration".

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{(2k+1)!} \int x^{4k+2}\,dx\right] = \left[\frac{x^3}{3}\right] - \left[\frac{x^7}{7 \cdot 3!}\right] + \left[\frac{x^{11}}{11 \cdot 5!}\right] - \cdots$$

$$\boxed{\int \sin x^2\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!}} = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots$$

Euler's Identity.

COMPLEX NUMBER is of the form $a+bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

Notes:

$$\left.\begin{aligned} i^2 &= -1 \\ i^3 &= i^2(i) = -i \\ i^4 &= i^3(i) = (-i)(i) = 1 \\ i^5 &= i^4(i) = i \end{aligned}\right\} \text{cyclic pattern}$$

$$i^6 = i^5(i) = i(i) = -1.$$

Let's substitute $x = i\theta$, $\theta \in \mathbb{R}$, into the Taylor series for e^x:

this should make you nervous — we have not defined exponentiat'n for imag. #'s !!

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots$$

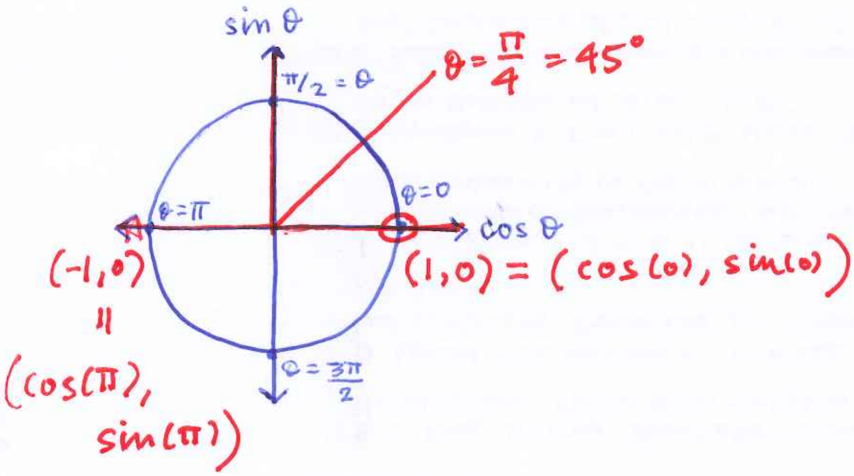
$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}$$

$$= \cos(\theta) + i\sin(\theta).$$

So, $\boxed{\text{DEFINE: } e^{i\theta} := \cos(\theta) + i\sin(\theta)}$ ← EULER'S IDENTITY

Euler's identity: $e^{i\theta} := \cos(\theta) + i\sin(\theta)$.

Plug in $\theta = \pi$: $e^{i\pi} = \cos(\pi) + i\sin(\pi)$



$$= -1 + i(0)$$

$$= -1$$

Therefore, $e^{i\pi} = -1$ or $\boxed{e^{i\pi} + 1 = 0}$

The 5 most important constants in math!
(maybe)