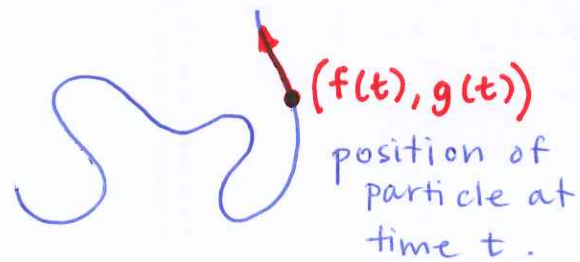
# Lecture 7: Parametric Curves (11.1, 11.2)



**Announcements:**

- HW4 due ~~Monday~~ WEDNESDAY, August 5 11:59 p.m.
- Midterm grades on myWPI ... see announcement there about final exam "deal", but the point: If you do better on the final than on the midterm, then your final grade replaces your midterm grade.

## 11.1: Parametrizations of Plane Curves.




- This particle path is not the graph of a function of the variable $x$ (Why?)
- But sometimes, can describe the path as $x = f(t)$, $y = g(t)$, for $f$ and $g$ cts. fns. — at time $t$, $(x,y) = (f(t), g(t))$.

**DEF.** If $x$ and $y$ are given as functions $x = f(t)$, $y = g(t)$ over an interval $I$ of $t$-values, then the set of points $(x,y) = (f(t), g(t))$ defined by these eq'ns is a PARAMETRIC CURVE. The eq'ns are PARAMETRIC EQUATIONS for the curve.

- The variable $t$ is called a PARAMETER for the curve
- $I$ is the PARAMETER INTERVAL. If $I = [a, b]$ (i.e., if we have $a \le t \le b$) then $(f(a), g(a))$ is the INITIAL POINT of the curve and $(f(b), g(b))$ is the TERMINAL POINT.

- The parametric eq'ns together with the interval $I$ constitute a <u>parametrization</u> of the curve, and when we give a parametrization, we say we have <u>parametrized</u> the curve.

| EXAMPLE 1, p. 654 |
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Sketch the curve defined by the parametric eq'ns:

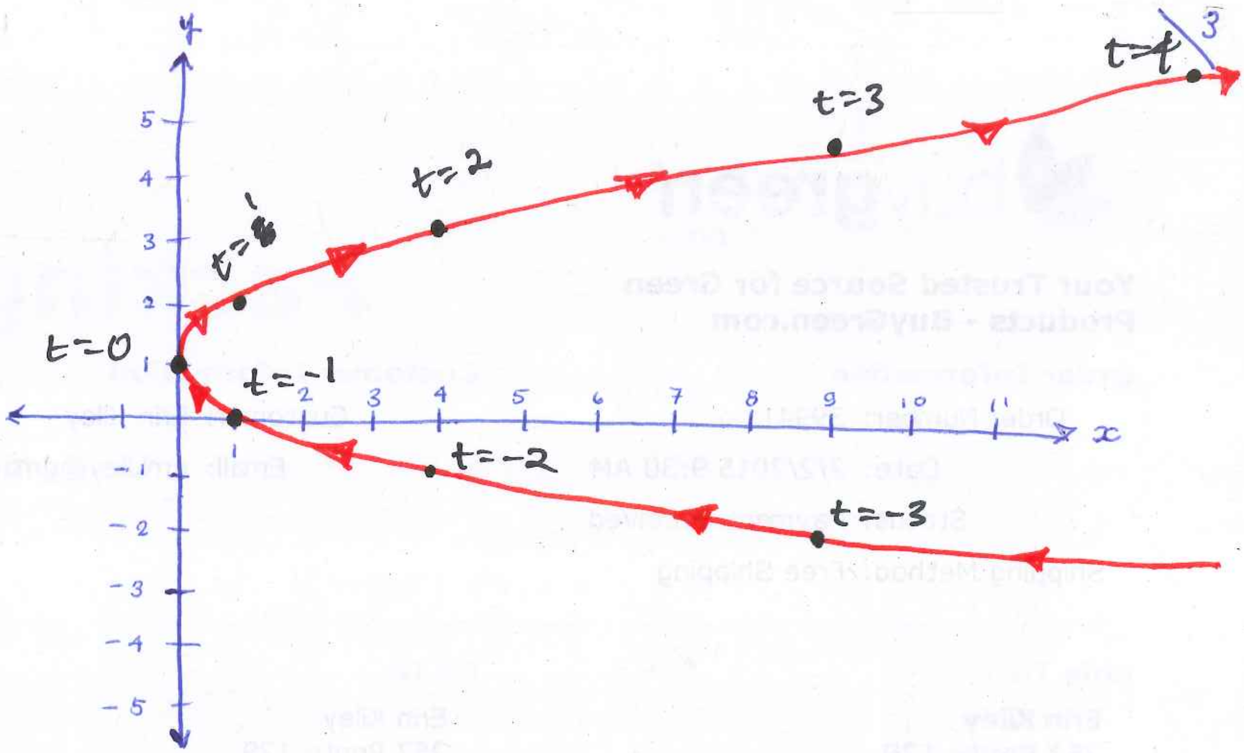
$$x = t^2 \quad , \quad y = t + 1 \quad , \quad -\infty < t < \infty .$$

Each value of $t$ gives a point $(x, y)$ on the parametriyed curve, so we just compute a list of values, plot them and draw a smooth curve connecting them:

| $t$ | $x = t^2$ | $y = t + 1$ | |
| --- | --- | --- | --- |
| $-5$ | $25$ | $-4$ | $(25, -4)$ |
| $-3$ | $9$ | $-2$ | $(9, -2)$ ✓ |
| $-2$ | $4$ | $-1$ | $(4, -1)$ ✓ |
| $-1$ | $1$ | $0$ | $(1, 0)$ ✓ |
| $0$ | $0$ | $1$ | $(0, 1)$ ✓ |
| $1$ | $1$ | $2$ | $(1, 2)$ ✓ |
| $2$ | $4$ | $3$ | $(4, 3)$ ✓ |
| $3$ | $9$ | $4$ | $(9, 4)$ ✓ |
| $4$ | $16$ | $5$ | $(16, 5)$ ✓ |
| $5$ | $25$ | $6$ | $(25, 6)$ |

**EXAMPLE 1, ct'd.**




- Think of this curve as the path of a moving particle — then we can draw arrows in the direc'n from lower t-values to higher t-values.
- Particle covers the same amount of <u>time</u> btwn. each pair of successive points (we chose $t = -3, -2, -1, 0, 1, 2, 3$ with <u>equal spacing</u>) — but NOT the same amt. of distance. When is it travelling faster?

**EXAMPLE 2, p. 654** Eliminate the parameter $t$ and <u>identify</u> the curve.

Well, $x = t^2$ and $y = t + 1$. Solve for t in terms of y: $t = y - 1$, so that $x = t^2 = (y-1)^2 = y^2 - 2y + 1$. This is the eq'n of a <u>parabola</u> in $y$, which the graph confirms.

$$x = y^2 - 2y + 1$$

Alt.: $t = \pm\sqrt{x}$

$y = \pm\sqrt{x} + 1$

$\boxed{\text{EXAMPLE 3, p. 654}}$ Graph the parametric curves:

(a) $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$

(b) $x = a\cos(t)$, $y = a\sin(t)$, $0 \le t \le 2\pi$

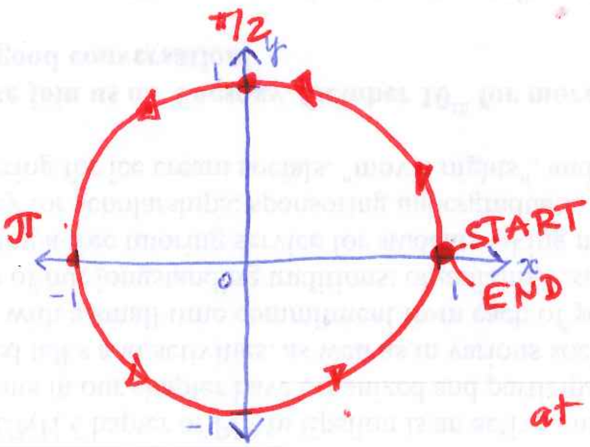
$(x-a)^2 + (y-b)^2 = r^2$

Familiar?

$\boxed{x^2 + y^2 = 1}$

(a) $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$.

So this is the graph of a CIRCLE centered at the origin (0,0) with radius 1.




at $t=0$, $x(0) = \cos(0) = 1$

$y(0) = \sin(0) = 0$

$(1,0)$

at $t = 2\pi$, $x(2\pi) = 1$

$y(2\pi) = 0$

Where does the particle start from?

Where does it stop?

In which direction does it traverse the circle?

(b) Similar — the only difference is radius = a.

| EXAMPLE |
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| 4, p.655 |

The position $P(x,y)$ of a particle moving in the $xy$-plane is given by the equations and parameter interval:

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

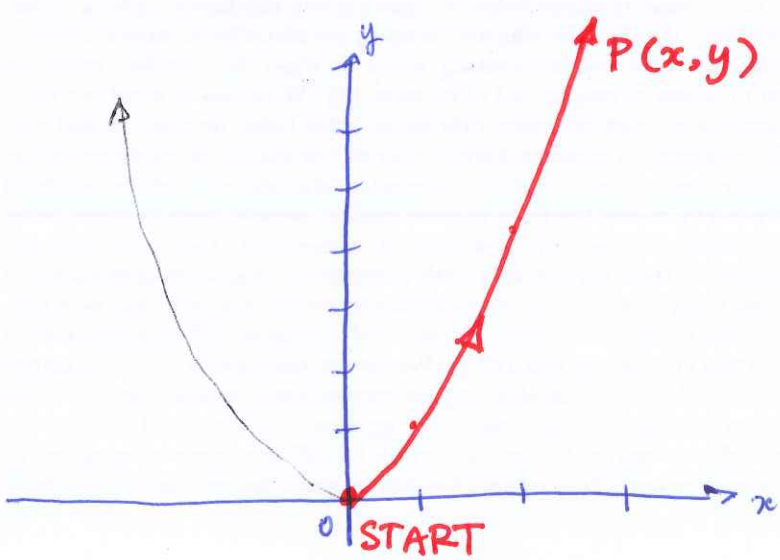
Well, $x = \sqrt{t}$ implies $x^2 = t$, so $y = x^2$ for every point $P(x,y)$ on the particle's path.

This is a parabola.

BUT!

The function $\sqrt{\cdot}$ is defined as the positive root — so the $x$-coords are never negative.

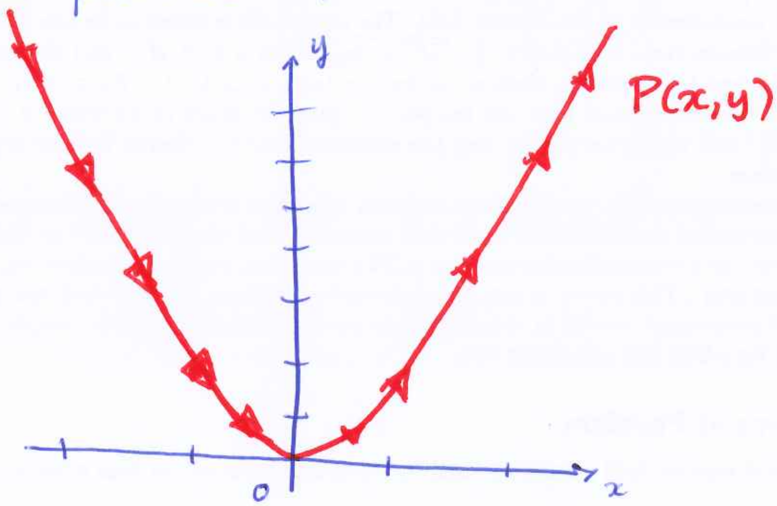
Also, beware of the interval $t \geq 0$:



**EXAMPLE 5, p. 655** But consider

$$x = t \quad , \quad y = t^2 \quad , \quad -\infty < t < \infty .$$

Easy — $y = x^2$ again, but as $t \in (-\infty, \infty)$, we trace out the whole parabola:




- What are the starting & terminal points?

  We have none.

- Note: The graph of any function $y = f(x)$ has the "NATURAL PARAMETRIZATION" $x = t$ , $y = f(t)$ , with $t \in \text{DOMAIN}\{f(x)\}$

**EXAMPLE 6, p. 655** Find a parametrization of the line through $(a,b)$ with slope $m$.

Cartesian Eq'n : $y - b = m(x - a) \Leftrightarrow y = mx + (b - ma)$

Natural parametrization: $\boxed{x = t \quad , \quad y = mt + (b - ma) , \; t \in (-\infty, \infty)}$

Another parametrization: $t = x - a \Rightarrow \boxed{x = t + a \quad , \quad y = mt + b \; , -\infty < t < \infty}$

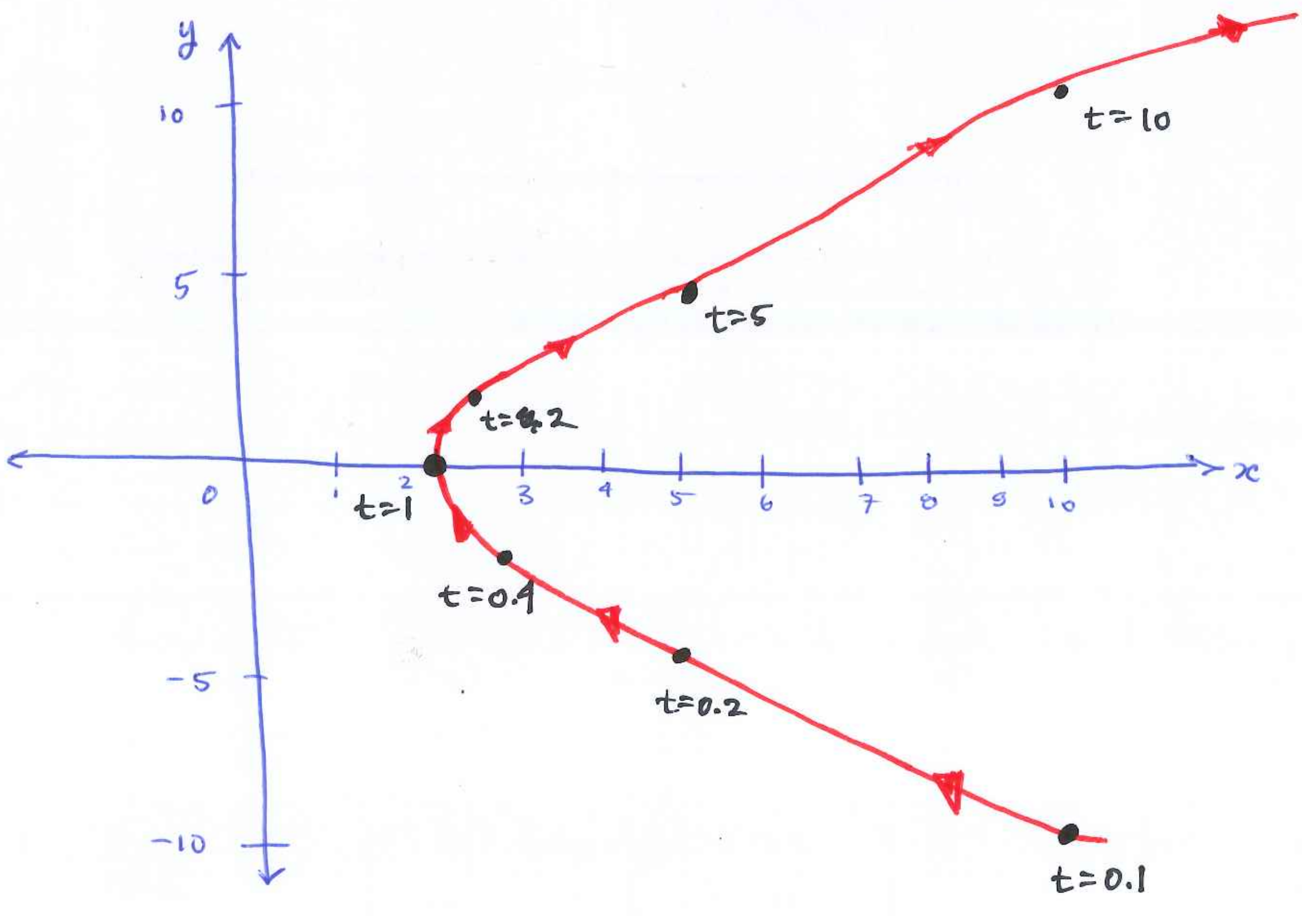
**EXAMPLE**
8, p. 656

Sketch & identify the path traced by $P(x,y)$ if:

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

Table.

| $t$ | $1/t$ | $x = t + \frac{1}{t}$ | $y = t - \frac{1}{t}$ | |
|---|---|---|---|---|
| 0.1 | 10.0 | 10.1 | −9.9 | $(10.1, -9.9)$ @ $t = 0.1$ |
| 0.2 | 5.0 | 5.2 | −4.8 | $(5.2, -4.8)$ @ $t = 0.2$ |
| 0.4 | 2.5 | 2.9 | −2.1 | $(2.9, -2.1)$ @ $t = 0.4$ |
| 1.0 | 1.0 | 2.0 | 0.0 | $(2, 0)$ @ $t = 1$ |
| 2.0 | 0.5 | 2.5 | 1.5 | $(2.5, 1.5)$ @ $t = \cancel{0.5}\ 2$ |
| 5.0 | 0.2 | 5.2 | 4.8 | $(5.2, 4.8)$ @ $t = 5$ |
| 10.0 | 0.1 | 10.1 | 9.9 | $(10.1, 9.9)$ @ $t = 10$ |



EXAMPLE 8, p. 656, ct'd

How to eliminate the parameter $t$?

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0$$

$$\underbrace{x + y}_{2t} = t + \frac{1}{t} + \left(t - \frac{1}{t}\right) \qquad \underbrace{x - y}_{\frac{2}{t}} = t + \frac{1}{t} - \left(t - \frac{1}{t}\right)$$

So $(x+y)(x-y) = (2\cancel{t})\left(\frac{2}{\cancel{t}}\right) = 4$,

i.e., $x^2 - y^2 = 4$,

which is the eq'n of a hyperbola... BUT

still, the $x$-coordinate of OUR graph must

be positive (WHY? — $x = t + \frac{1}{t}$, $t > 0$).

* Be careful about the parameter interval!

Questions?

(Cool reading abt. cycloids & brachistochrones on p. 657)

## 11.2: Calculus w/ parametric curves.

$a = b \cdot c$

$b = \frac{a}{c}$

• DERIVATIVES

Chain rule (recall): $\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$,

so $$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

if all 3 derivs exist and $\frac{dx}{dt} \neq 0$.

• SECOND DERIVATIVES

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\cancel{\frac{d}{dt}}\left[\frac{dx}{dt}\right]}$$

**EXAMPLE 1, p. 661** Find the tangent to the curve
$x = \sec(t)$, $y = \tan(t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$
at the point $(\sqrt{2}, 1)$ where $t = \pi/4$.

$$\frac{dx}{dt} = \frac{d}{dt}[\sec(t)] = \frac{d}{dt}\left[\frac{1}{\cos(t)}\right] = \frac{\sin(t)}{\cos^2(t)} = \tan(t)\sec(t)$$

$$\frac{dy}{dt} = \frac{d}{dt}[\tan(t)] = \frac{d}{dt}\left[\frac{\sin(t)}{\cos(t)}\right] = \frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)} = \sec^2(t)$$

So $\frac{dy}{dx} = \frac{\sec^2(t)}{\tan(t)\sec(t)} = \frac{\sec(t)}{\tan(t)} = \frac{\cos(t)}{\cos(t)\sin(t)} = \frac{1}{\sin(t)} = \csc(t)$, so $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{1}{\sin(\frac{\pi}{4})} = \sqrt{2}$

and the line: $y - 1 = \sqrt{2}\,(x - \sqrt{2}) \Rightarrow \boxed{y = \sqrt{2}\,x - 1}$

EXAMPLE 2, p. 662

Find $\frac{d^2y}{dx^2}$ as a fn. of $t$ if: $x = t - t^2$, $y = t - t^3$.

(1) Express $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ in terms of $t$.

$$\frac{dy}{dt} = \frac{d}{dt}\left[t - t^3\right] = 1 - 3t^2$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[t - t^2\right] = 1 - 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

(2) Differentiate the resulting $\frac{dy}{dx}$ w.r.t. $t$. (with respect to)

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{1-3t^2}{1-2t}\right] = \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1-2t)^2} = \frac{6t^2 - 6t + 2}{(1-2t)^2}$$

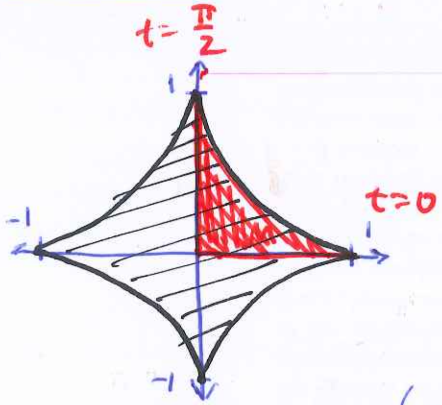
(3) Divide the resulting $\frac{d^2y}{dt\,dx}$ by $\frac{dx}{dt}$:

$$\frac{d^2y}{dx^2} = \frac{d^2y/dtdx}{dx/dt} = \frac{6t^2 - 6t + 2 \Big/ (1-2t)^2}{1-2t}$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^3}.$$

**EXAMPLE 3, p. 662** Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \boxed{\sin^3(t)}, \quad 0 \le t \le 2\pi.$$




Note: the astroid is symmetric, so the total area is 4 times the area beneath the curve in the first quadrant.

(The 1st quadrant has $t \in [0, \pi/2]$.)

Calculate this area by integrating:

$$A = 4 \int_{x=0}^{1} y \, dx$$

$$= 4 \int_{t=0}^{\pi/2} \sin^3(t) \cdot \underbrace{\left[ \frac{dx}{dt} \, dt \right]}$$

$$= \frac{d}{dt}\left[\cos^3(t)\right] \cdot dt$$

$$= \left[-3 \sin(t) \cos^2(t)\right] dt$$

$$= 4 \int_{t=0}^{\pi/2} \sin^3(t) \cdot (-3 \sin(t) \cos^2(t)) \, dt$$

$$= -12 \int_0^{\pi/2} \sin^4(t) \cos^2(t) \, dt$$

⋮ many steps later (see p. 663)

$$= \boxed{3\pi/8}$$

- LENGTH OF PARAMETRICALLY DEFINED CURVE

**DEF.** If $C$ is defined parametrically by $x = f(t)$ and $y = g(t)$, $t \in [a,b]$, where $f'(t)$ and $g'(t)$ are cts 1 and NOT simultaneously 2 zero on $[a,b]$, and $C$ is traversed exactly 3 once as $t$ increases from $t=a$ to $t=b$, then the LENGTH OF $C$ is:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

(derivation on p.663)

EXAMPLE 4, p.664

Find the length of the circle of radius $r$:

$$x = r\cos(t), \quad y = r\sin(t), \quad 0 \le t \le 2\pi.$$

Check: $f'(t) = -r\sin(t)$, $g'(t) = r\cos(t)$ — both cts.? 1 ✓ Not simultaneously 0? 2 ✓

Is $C$ traversed exactly once? 3 ✓

$$L = \int_0^{2\pi} \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\, dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt =$$

$$= \int_0^{2\pi} \sqrt{r^2 \underbrace{(\sin^2 t + \cos^2 t)}_{=1}}\, dt = \int_0^{2\pi} r\, dt = rt \Big|_0^{2\pi} = \boxed{2\pi r.}$$

| EXAMPLE |
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| 5, p. 665 |

Find the length of the astroid

$$x = \cos^3(t) \quad , \quad y = \sin^3(t) \quad , \quad 0 \le t \le 2\pi .$$

Exploit symmetry again

| EXAMPLE |
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| 6, p. 665 |

Find the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Parametrically, choose $x = a \sin(t)$, $y = b \cos(t)$, $0 \le t \le 2\pi$.

* SEE p. 666 FOR THE INTEGRAL.

• AREA OF SURFACE OF REVOLUTION

If a smooth curve $x = f(t)$, $y = g(t)$, $t \in [a,b]$ is traversed exactly once as $t$ increases from $a$ to $b$, then the areas of the surfaces generated by revolving the curve about the coordinate axes:

(1) Revol'n abt. $x$-axis $(y \geq 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(2) Revol'n abt. $y$-axis $(x \geq 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

| EXAMPLE |
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| 9, p. 669 |

Unit circle: center $(0,1)$ $\quad x = \cos(t)$, $y = 1 + \sin(t)$, $t \in [0, 2\pi]$.

Find the area of surface swept out by revolving circle abt. $x$-axis.