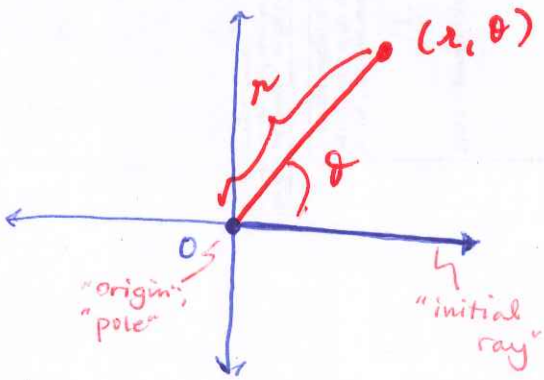
Lecture 8: Polar coordinates (11.3 – 11.5)

Announcements.

- HW4 due Wednesday, 11:59 p.m.
- HW5 due Monday, 11:59 p.m.
- Sample problems on myWPI – from old textbook, but relevant material! Take a look!

11.3: Polar Coordinates.



- Polar coordinates are not unique!

EX. $P(2, \pi/6) = P(2, 13\pi/6) = P(2, 25\pi/6) = \ldots$

$= P(-2, 7\pi/6) = P(-2, 19\pi/6) = \ldots$

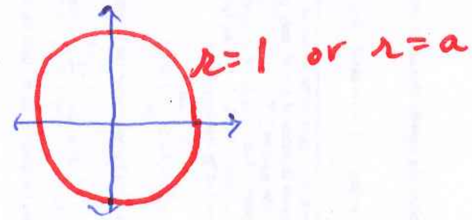
So
$$\left.\begin{aligned} P(2, \pi/6) &= P(2, \pi/6 \pm 2n\pi) \\ &= P(-2, -5\pi/6 \pm 2n\pi) \end{aligned}\right\} n \in \mathbb{N} \cup \{0\}$$

EXAMPLES.

- Graph of a circle centered at the origin

$r = 1$

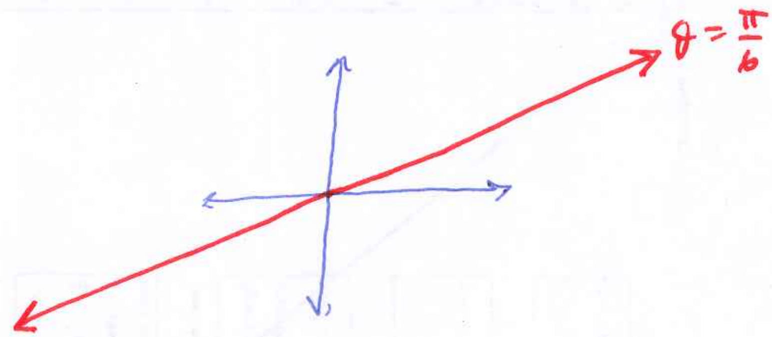
$r = a$



- Graph of a line through the origin

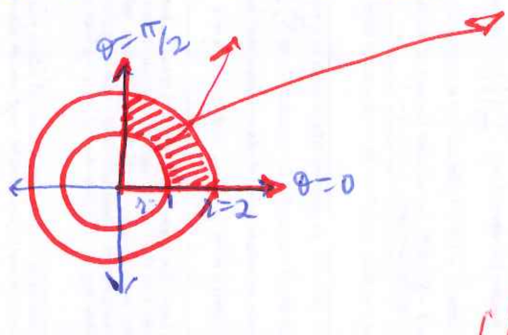
$\theta = \frac{\pi}{6}$

$\theta = \alpha$



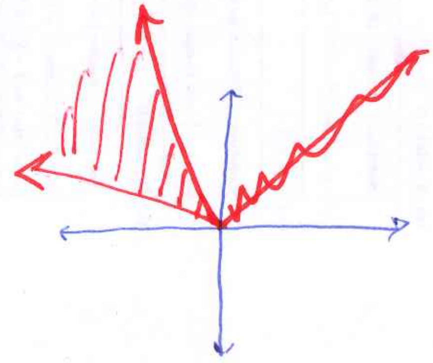
- Combinations of the above

  - $\{(r, \theta) : 1 \leq r \leq 2 \text{ and } 0 \leq \theta \leq \frac{\pi}{2}\}$



$\{x : \_\_\_\_\}$

  - $\{(r, \theta) : \frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}\}$



• Relating Polar & Cartesian coordinates.

$$\boxed{\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}}$$

← "UNIT" CIRCLE - RECALL

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) \\ &= r^2 \end{aligned}$$

$$\begin{aligned} \frac{y}{x} &= \frac{r \cdot \sin(\theta)}{r \cdot \cos(\theta)} \\ &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \tan(\theta) \end{aligned}$$

so

$$\boxed{\begin{aligned} r^2 &= x^2 + y^2 \\ \tan(\theta) &= \frac{y}{x} \end{aligned}}$$

$\boxed{\text{EXAMPLES} \quad 4, \text{p. } 673}$

(a) Polar: $r\cos(\theta) = 2$

$x = 2$

(b) Cartesian: $xy = 4 \Rightarrow (r\cos(\theta))(r\sin\theta) = 4$

$$\boxed{r^2(\sin\theta\cos\theta) = 4}$$

$$r^2 = \frac{4}{\sin\theta\cos\theta} = 4\sec\theta\csc\theta$$

(c) Cartesian: $x^2 - y^2 = 1$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1$$

(d) Polar: $r = 1 + 2r\cos(\theta)$

$r = \pm\sqrt{x^2+y^2}$

$r\cos\theta = x$

$$\pm\sqrt{x^2+y^2} = 1 + 2x$$

$$x^2+y^2 = 1+4x+4x^2$$

$$y^2 = 1+4x+3x^2$$

(e) Polar: $r = 1 - \cos(\theta)$

Cartesian: $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

EXAMPLE 5, p. 673 Find a polar eq'n for the circle of radius 3, centered at $(0,3)$.

Cartesian: $x^2 + (y-3)^2 = 9$

simplify: $x^2 + y^2 - 6y + 9 = 9$

$$\underbrace{x^2 + y^2}_{r^2} - 6\underbrace{y}_{} = 0$$

$x = r \cdot \cos(\theta)$

$(y = r \cdot \sin(\theta))$

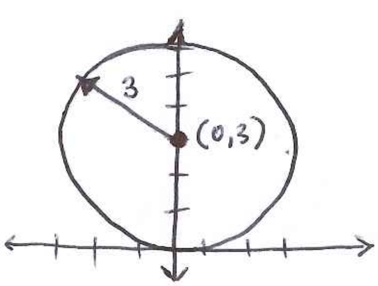
$(x^2 + y^2 = r^2)$

$$\boxed{r^2 - 6r\sin\theta = 0}$$

$$r(r - 6\sin\theta) = 0$$

$r = 0$ or $r = 6\sin\theta$

and note, $\boxed{r = 6\sin\theta}$ does include the possibility $r = 0$ (this occurs at $\theta = 0$).



EXAMPLES
6, p. 673

Replace the polar eq'ns by Cartesian ones:

$x = r\cos(\theta)$
$y = r\sin(\theta)$
$x^2+y^2=r^2$

(a) $r \cdot \cos(\theta) = -4$

$x = -4$ vertical line

(b) $r^2 = 4r\cos(\theta)$

$x^2+y^2 = 4x$

$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$

$(x-2)^2 + y^2 = 2^2$

COMPLETING THE SQUARE.

circle w/ radius 2 centered at (2,0)

(c) $r = \dfrac{4}{2\cos(\theta) - \sin(\theta)}$

$r(2\cos\theta - \sin\theta) = 4$

$2r\cos\theta - r\sin\theta = 4$

$2x - y = 4$

$y = 2x - 4$ line

11.4: Graphing eq'ns with polar coordinates.

Not always obvious what the graphs look like ...
(but also for Cartesian graphs)

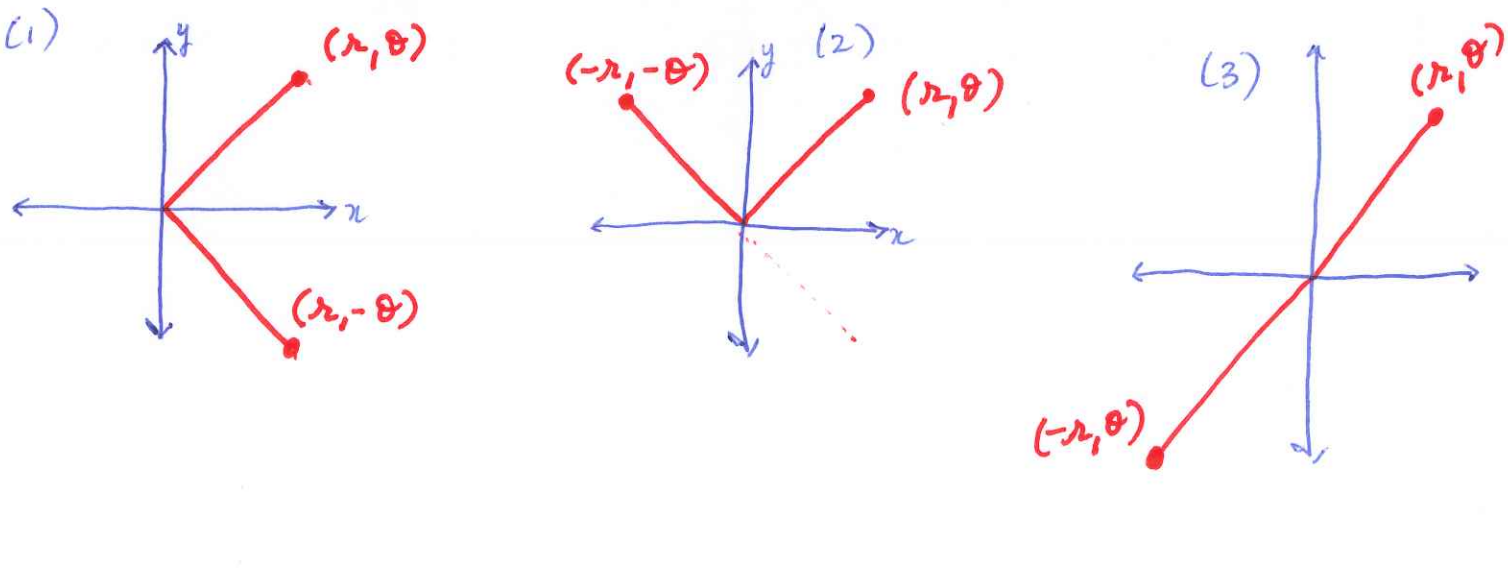
Some tricks for developing intuition!

- SYMMETRY.

(1) About x-axis: $\underset{\text{on graph}}{(r,\theta)} \overset{\text{"implies"}}{\Rightarrow} \underbrace{(r,-\theta) \text{ or } (-r, \pi-\theta)}_{\text{also on graph}}$

(2) About y-axis: $(r,\theta) \Rightarrow (-r,-\theta)$ or $(r, \pi-\theta)$

(3) Abt. origin: $(r,\theta) \Rightarrow (-r, \theta)$ or $(r, \theta+\pi)$



- SLOPE

If the polar eq'n is $r = f(\theta)$, then

$$\boxed{\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}}$$

(provided $\left.\frac{dx}{d\theta}\right|_{(r,\theta)} \neq 0$) – see p. 676 for derivation.

Suppose $r = f(\theta)$ passes through the origin when $\theta = \theta_0$.

Then $f(\theta_0) = 0$, so $\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{\cancel{f'(\theta_0)}\sin(\theta_0)}{\cancel{f'(\theta_0)}\cos(\theta_0)} = \tan(\theta_0)$.

- TABLE OF VALUES

Polar coordinate equations are like parametric eq'ns?

$x = r\cos(\theta) = f(\theta)\cos(\theta)$, $y = r\sin\theta = f(\theta)\sin(\theta)$

- PERIODICITY

Does the graph repeat itself? sin, cos have a period of 2π... for some eq'ns this is helpful.

EXAMPLE 1, p. 676 Graph the curve $r = 1 - \cos\theta$ in the coordinate plane.

SYMMETRY?

Notice that if (r,θ) is on the graph, then:

$$r = 1 - \cos\theta$$

"cos is an even function"

and since $\cos(-\theta) = \cos(\theta)$, this means that also,

$$r = 1 - \cos(-\theta),$$

so $(r, -\theta)$ should be on the graph.

The graph is therefore symmetric about the <u>x-axis</u>.

SLOPE AT ORIGIN

Consider that $r = 0$ when $1 - \cos\theta = 0$, i.e., when $\cos\theta = 1$, so for $\theta_0 = \underline{0 + 2\pi m}$ ($m \in \mathbb{Z}$). At this point, the slope of the graph is ~~equal to~~ given by: $\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \tan(\theta_0) = \tan(0) = \frac{\sin(0)}{\cos(0)} = 0$.

Horizontal tan line at origin.

PERIODICITY?

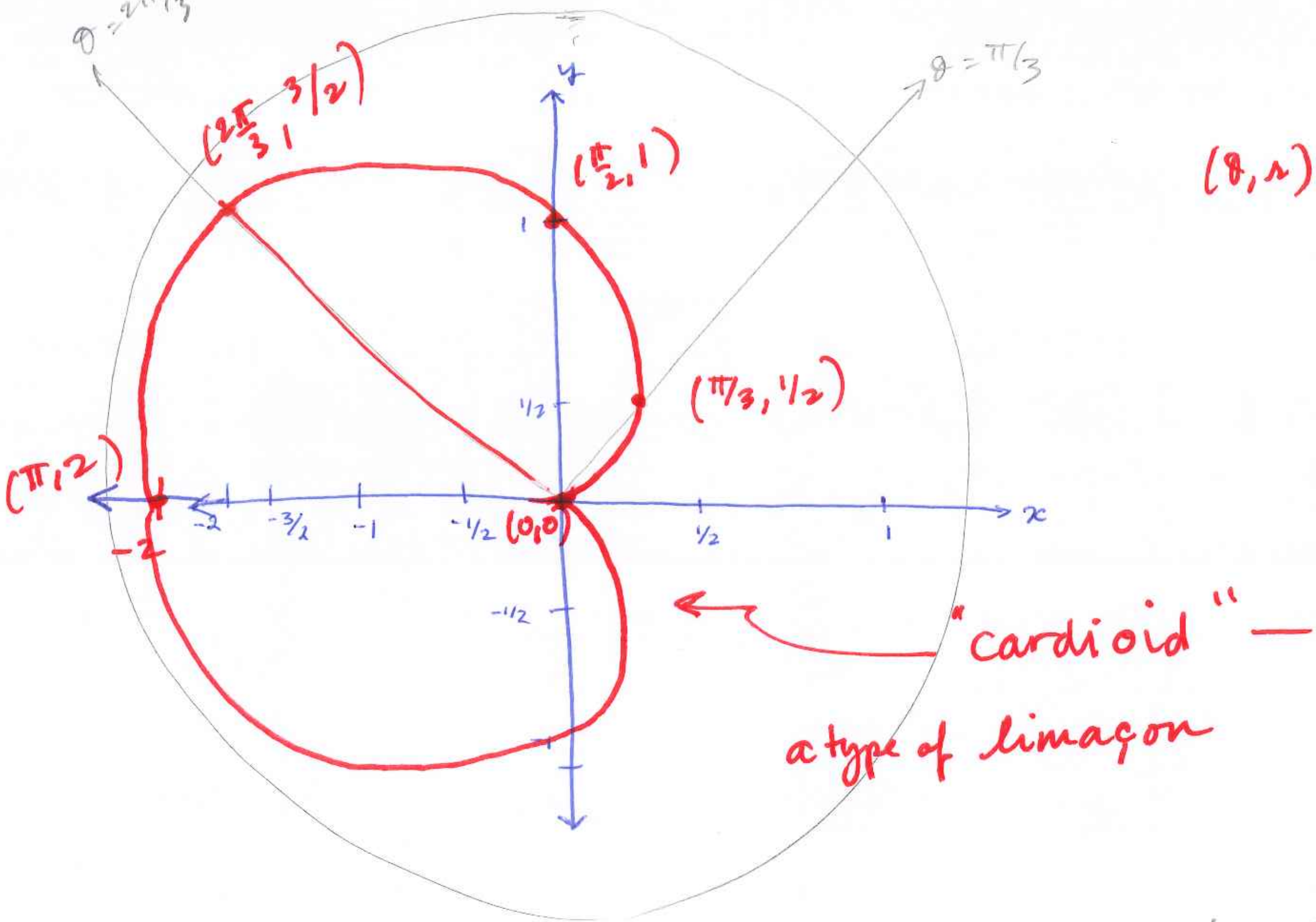
Notice that $1 - \cos(\theta)$ is periodic – that is, ~~[illegible]~~ for all θ, $1 - \cos(\theta) = 1 - \cos(\theta + 2\pi)$, since cos is 2π-periodic. [Aside: if it had been $\theta - \cos(\theta)$, <u>no longer periodic!</u>]. So only have to worry abt. $\theta \in [0, 2\pi]$.

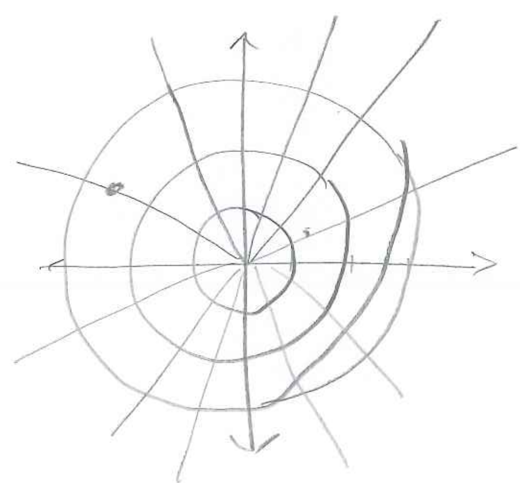
But because of symmetry, only have to worry abt. $\theta \in [0, \pi]$:

θ	0	$\frac{\pi}{4}$	$\pi/3$	$\pi/2$	$2\pi/3$	π
$r = 1 - \cos\theta$	0	$\frac{2-\sqrt{2}}{2}$	$1/2$	1	$3/2$	2

EXAMPLE 1, ct'd.

So, the graph:





$\boxed{\text{EXAMPLE 2, p. 677}}$ Graph the curve $r^2 = 4\cos(\theta)$ in the xy-plane.

- SYMMETRY?

(r,θ) on graph $\Rightarrow r^2 = 4\cos(\theta)$

$\Rightarrow (-r)^2 = 4\cos(\theta)$ as $r^2 = (-r)^2$

so $(-r,\theta)$ on graph

$\Rightarrow$ symmetry about the origin

$\Rightarrow r^2 = 4\cos(-\theta)$ as $\cos(\theta) = \cos(-\theta)$

so $(r,-\theta)$ on graph

$\Rightarrow$ symmetry about the x-axis

$\Rightarrow (-r)^2 = 4\cos(-\theta)$ as both $(-r)^2 = r^2$ and $\cos(-\theta) = \cos(\theta)$

so $(-r,-\theta)$ on graph

$\Rightarrow$ symmetry about the y-axis

- SLOPE?

$r = 0$ when $4\cos(\theta) = 0$, i.e., when $\theta = \pm\frac{\pi}{2}$. Here, $\left.\frac{dy}{dx}\right|_{(0,\pm\frac{\pi}{2})} = \tan\left(\pm\frac{\pi}{2}\right) \searrow \infty$

so ~~slope of~~ the tan. line is vertical at origin.

- PERIODICITY?

Graph is 2π-periodic, since $4\cos(\theta) = 4\cos(\theta + 2\pi)$, $\forall\theta$.

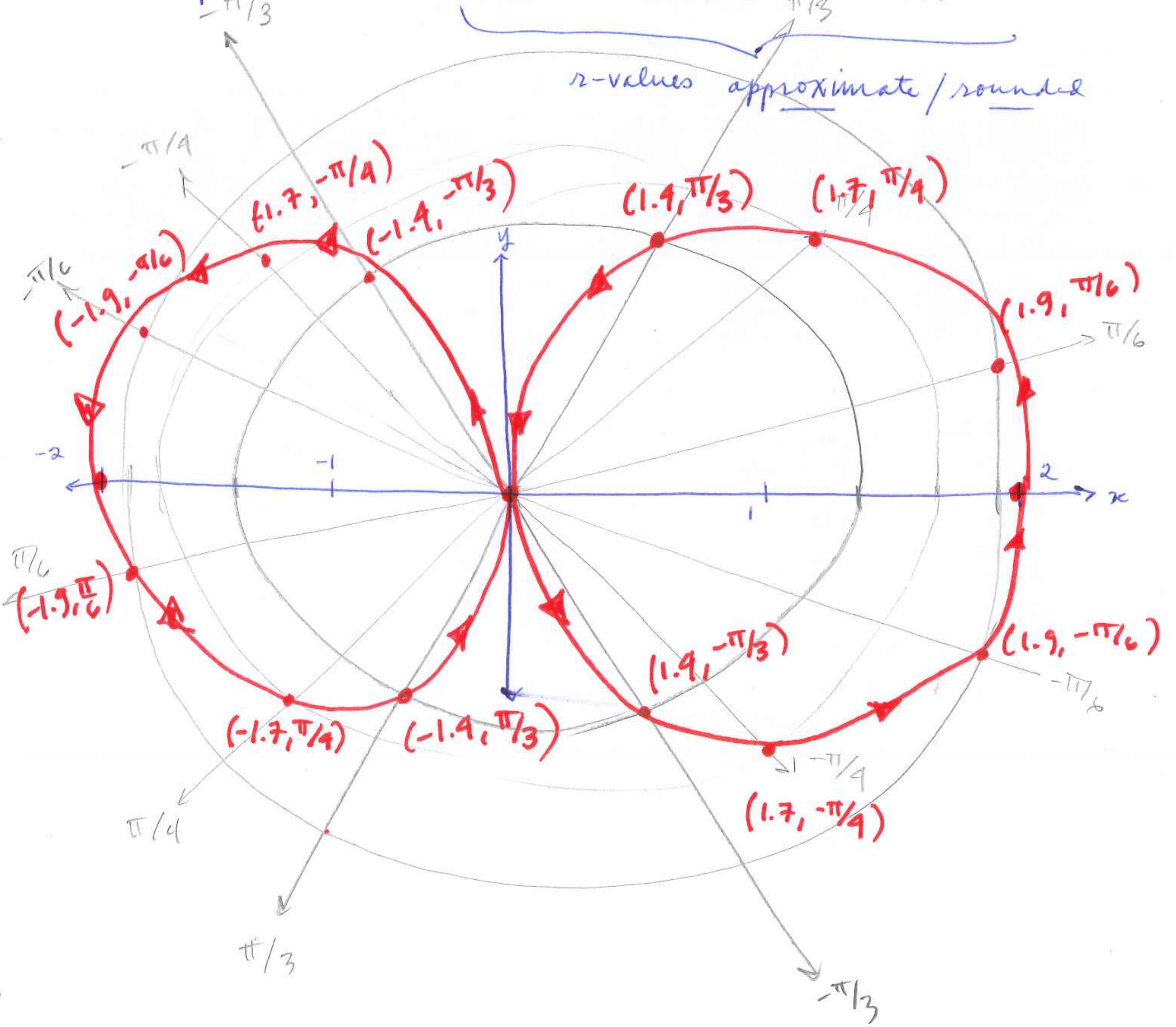
So we make a table of values for $\theta \in [-\pi/2, \pi/2]$, because of symmetry + periodicity, and we connect using a vertical tan. line at origin.

EXAMPLE 2, ct'd.

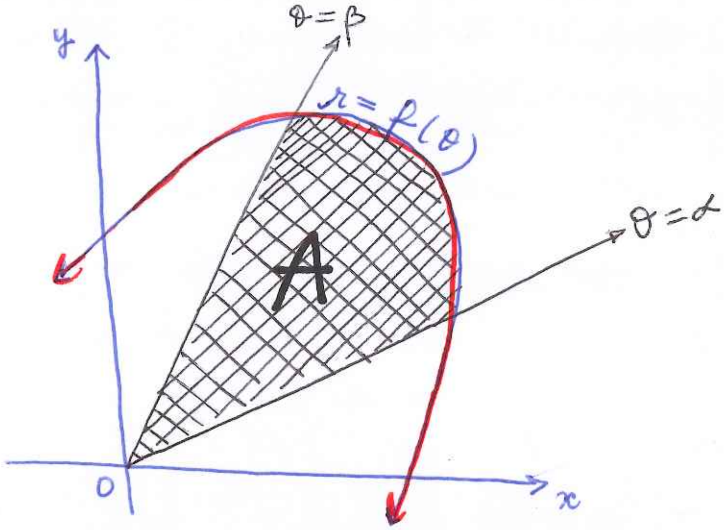
$$r^2 = 4\cos\theta$$

θ	0	$\pm\pi/6$	$\pm\pi/4$	$\pm\pi/3$	$\pm\pi/2$
$r = \pm 2\sqrt{\cos\theta}$	± 2	± 1.9	± 1.7	± 1.4	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxx}}_{r\text{-values approximate / rounded}}$



11.5 : Areas + Lengths in Polar Coordinates.



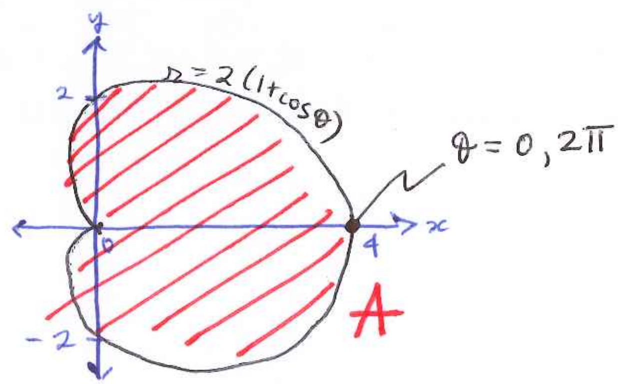
$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta}$$

(Derivation, p. 679)

EXAMPLE 1, p. 680

Find the area of the region in the xy-plane bounded by the cardioid

$$r = 2(1+\cos(\theta)).$$



What is the θ-interval? – Determine by graphing that $\theta \in [0, 2\pi]$ sweeps out the cardioid exactly once, so:

$$A = \int_0^{2\pi} \frac{1}{2}\left[2(1+\cos\theta)\right]^2 d\theta$$

$$= \int_0^{2\pi} 2 + 4\cos(\theta) + \underbrace{2\cos^2(\theta)} \, d\theta \qquad \boxed{\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))}$$

$$= \int_0^{2\pi} 2 + 4\cos(\theta) + 1 + \cos(2\theta) \, d\theta$$

$$= \int_0^{2\pi} 3 + 4\cos(\theta) + \cos(2\theta) \, d\theta$$

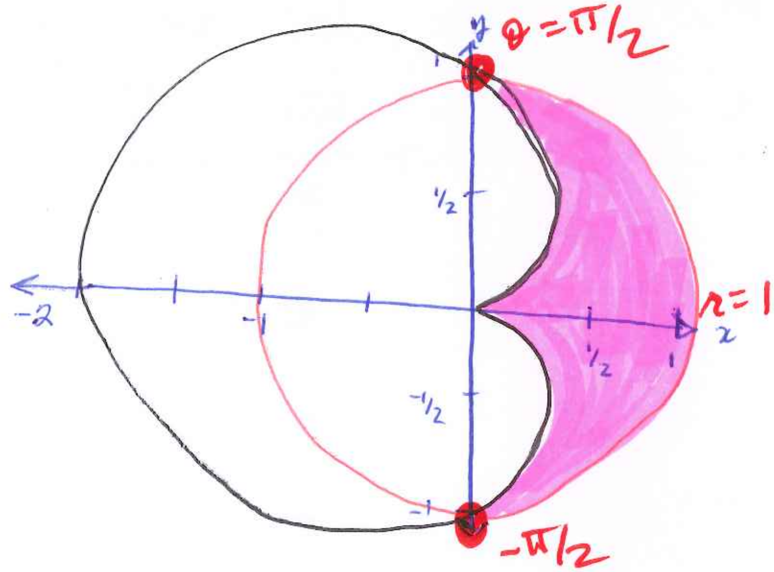
$$= 3\theta + 4\sin(\theta) + \frac{1}{2}\sin(2\theta) \Big|_0^{2\pi}$$

$$= 3(2\pi) + \cancelto{0}{4\sin(2\pi)} + \cancelto{0}{\frac{1}{2}\sin(4\pi)} - \cancelto{0}{3(0)} - \cancelto{0}{4\sin(0)} - \cancelto{0}{\frac{1}{2}\sin(0)} = \boxed{6\pi}$$

EXAMPLE 2, p. 680 Find the area of the region that lies inside the circle $r=1$ and outside the cardioid $r = 1-\cos\theta$.

ALWAYS SKETCH GRAPH FIRST !

Recall – we already did $r = 1-\cos\theta$ today:



- Add the circle of radius 1.
- The area we want

So, we compute the area inside the circle on $[-\pi/2, \pi/2]$ and subtract the area inside the cardioid on $[-\pi/2, \pi/2]$.

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1)^2 - \frac{1}{2}(1-\cos(\theta))^2 \, d\theta$$

$$\boxed{\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))}$$

$$= \int_{-\pi/2}^{\pi/2} \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \cos(\theta) - \frac{1}{2}\cos^2(\theta)\, d\theta = \int_{-\pi/2}^{\pi/2} \cos(\theta) - \frac{1}{4} - \frac{1}{4}\cos(2\theta)\, d\theta =$$

even fn. over an interval symm. abt. origin

$$= 2\int_0^{\pi/2} -1/4 + \cos\theta - 1/4\cos(2\theta)\, d\theta = -\frac{1}{2}\theta + 2\sin\theta - \frac{1}{4}\sin(2\theta)\Big|_0^{\pi/2} =$$

$$= -\frac{1}{2}(\pi/2) + 2\underbrace{\sin(\pi/2)}_{1} - 1/4\underbrace{\sin(\pi)}_{0} + \underbrace{\frac{1}{2}(0)}_{0} - 2\underbrace{\sin(0)}_{0} + 1/4\underbrace{\sin(0)}_{0} = \boxed{2-\pi/4}$$

Length of a polar curve.

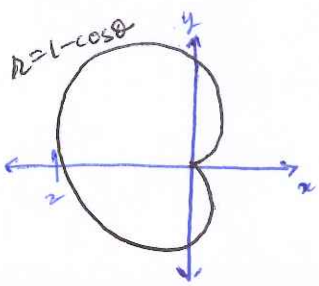
If $r = f(\theta)$ has cts. 1st deriv. for $\alpha \le \theta \le \beta$
and if the pt. $P(r,\theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β,
Then the length of the curve is:

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(derivation, p. 681)

EXAMPLE 3, p. 681 Find the length of the cardioid $r = 1 - \cos\theta$.

ALWAYS SKETCH FIRST !



So the cardioid is traced once for $\theta \in [0, 2\pi]$ and the length:

$$L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta =$$

$$= \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}}\, d\theta =$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{4\sin^2(\theta/2)}\, d\theta = \int_0^{2\pi} 2|\sin\theta/2|\, d\theta =$$

$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

$$= \int_0^{2\pi} 2\sin(\theta/2)\, d\theta = -4\cos(\theta/2)\Big|_0^{2\pi} = -4\cos\left(\frac{2\pi}{2}\right) + 4\cos\left(\frac{0}{2}\right)$$

$$= -4(-1) + 4(1) = \boxed{8}$$