Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due Friday, July 24, 2015, 11:59 p.m.

Week 2: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it.

Reading

Please read Sections 10.2 and 10.3 in time for Tuesday's lecture, and Section 10.4 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 10, "Questions to Guide Your Review", p. 647, Problems 6–16

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 10.2, Problems 1, 3, 7–17 odd, 19, 21, 23, 27–47 odd, 49–55 odd, 69–77 odd
- Section 10.3, Problems 1–39 odd, 43, 44
- Section 10.4, Problems 1–31 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due Friday, July 24, 2015, 11:59 p.m. EDT

Name: ______________________

WPI Username: ______________________

Week 2: Homework Problems

Due date: Friday, July 24, 2015, 11:59 p.m. EDT. Please upload a single `.pdf` document to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly.

III) **Show your work**. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. **[10 points]** For which values of x does the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\left(\frac{1}{3+\sin x}\right)^n$$

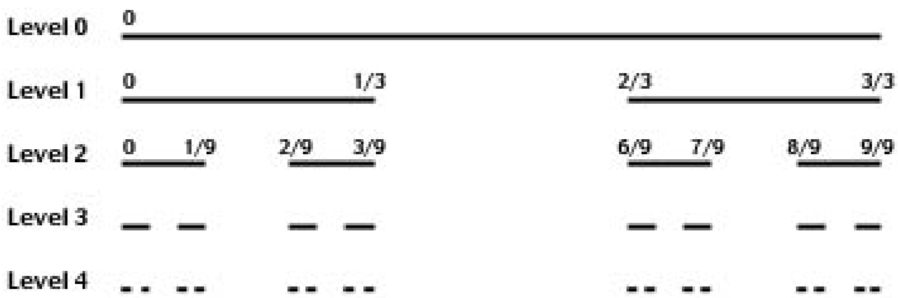
converge? What is the sum when the series does converge? [Hint: Express the inequality $|r|<1$ in terms of x. The sum should also necessarily involve x.]

Solution. This geometric series can be rewritten as $\displaystyle\sum_{n=0}^{\infty}\left(\frac{-1}{6+2\sin x}\right)^n$, which has $a=1$ and $r=\dfrac{-1}{6+2\sin x}$. The series will converge when $|r|<1$; that is, when $\left|\dfrac{-1}{6+2\sin x}\right|<1$, which occurs exactly when $|6+2\sin x|>1$. This last relation, however, is always true, since $|\sin x|<1$, so the geometric series converges for all x-values. It converges to $\dfrac{1}{1-r}$; that is,

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\left(\frac{1}{3+\sin x}\right)^n=\frac{1}{1+\frac{1}{6+2\sin x}}=\frac{6+2\sin x}{7+2\sin x}.$$

Problem 2. This problem explores the Cantor set. To construct this set, we begin with the closed interval $[0, 1]$. From that interval, remove the middle open interval $(1/3, -1/3)$, leaving two closed intervals $[0, 1/3]$ and $[2/3, 1]$. At the second step, remove the open middle third interval from each of those remaining—*i.e.*, from $[0, 1/3]$ we remove the open interval $(1/9, 2/9)$ and from $[2/3, 1]$ we remove $(7/9, 8/9)$, leaving behind the four closed intervals $[0, 1/9]$, $[2/9, 1]$, $[2/3, 7/9]$, and $[8/9, 1]$. At the next step, we remove the middle open third interval from each closed interval left behind, so $(1/27, 2/27)$ is removed from $[0, 1/9]$, leaving the closed intervals $[0, 1/27]$ and $[2/27, 1/9]$; $(7/27, 8/27)$ is removed from $[2/9, 7/27]$, leaving behind $[2/9, 7/27]$ and $[8/27, 1/3]$; and so forth.

We continue this process repeatedly without stopping, at each step removing the open third interval from the middle of ever closed interval remaining after the previous step. The numbers remaining in the interval $[0, 1]$, after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845–1918)[2]. A picture of this process is below. The set has some interesting properties.



(a) **[3 points]** The Cantor set contains infinitely many numbers in $[0, 1]$. List nine numbers that belong to the Cantor set.

Solution. The endpoints of the intervals that remain after each step will remain in the Cantor set; nine of these could be 0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, and 1/27.

(b) **[7 points]** At the first step, the length of the interval removed from $[0, 1]$ is $1/3$. At the second step, the total length of the intervals removed is $2 \cdot (1/9) = 2/9$. At the third step, the total length of the intervals removed is $4 \cdot (1/27) = 4/27$, and so forth. Write the correct geometric series that describes the total length of all the open middle third intervals that have been removed, and if it converges, find its sum.

Solution. The infinite series describing this total length is:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{2^{n-1}}{3^n} + \cdots = \sum_{0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n.$$

This is a geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$, and so since $|r| = \frac{2}{3} < 1$, it converges:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{2^{n-1}}{3^n} + \cdots = \sum_{0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = \frac{1}{3 - 2} = 1.$$

So, the Cantor set contains infinitely many numbers between 0 and 1, but in constructing it by removing certain pieces of the interval $[0, 1]$, we end up removing the entire length of the interval. This is one of many odd properties of the Cantor set!

[2]For more information about the Cantor set, including more pictures representing the various stages of its construction, please see `http://personal.bgsu.edu/~carother/cantor/Cantor1.html` , and remember to click the ‘more on the Cantor set’ link at the bottom of the page to continue reading.

Problem 3. Make up an infinite series of nonzero terms whose sum is...

(a) [**2 points**] 2.

Solution. Consider the geometric series $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$. This is a geometric series with $a = 1$ and $r = \frac{1}{2}$, and so since $|r| = 1/2 < 1$, the series converges to $\frac{1}{1-\frac{1}{2}} = 2$.

(b) [**1 point**] -4.

Solution. Consider the similar geometric series $\sum_{n=0}^{\infty} -2\left(\frac{1}{2}\right)^n$. This is a geometric series with $a = -2$ and $r = \frac{1}{2}$, and so since $|r| = 1/2 < 1$, the series converges to $\frac{-2}{1-\frac{1}{2}} = -4$.

(c) [**2 points**] 0.

Solution. Consider the series $S := 2 - \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$. The last part is a geometric series with $a = 1$ and $r = \frac{1}{2}$, and so since $|r| = 1/2 < 1$, the last part converges to $\frac{1}{1-\frac{1}{2}} = 2$. To obtain the sum of the entire series (that is, $2 - \sum(0.5)^n$), we subtract the sum of the convergent part from the sum of the finitely many (exactly one) leading terms: $S = 2 - 2 = 0$.

(d) [**5 points**] Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

Solution. In part (b), we made an infinite series of nonzero terms that converged to 0, so we know that this is possible. Also, consider the geometric series $\sum_{n=0}^{\infty}\frac{a}{2^n}$. This converges to $\frac{a}{1-\frac{1}{2}} = 2a$, and so if we want to create an infinite series of nonzero terms that converges to some $x \neq 0$, then we set $a := \frac{x}{2}$ and use the geometric series $\sum_{n=0}^{\infty}\frac{x}{2^{n+1}}$, which converges to x.

Problem 4. [**10 points**] As you know, **the harmonic series diverges** (the harmonic series diverges). But its partial sums just grow so, so slowly that there is no empirical evidence for this fact; suppose, in fact, that we had started computing the partial sums with $s_1 = 1$ the day the universe was formed, 13 billion years ago, and that we added a new term *every second*. About how large would the partial sum s_n be today, assuming that all years had 365 days, and that leap seconds (like the one we had earlier this year!) did not exist? **You must show your calculations.**

Solution. The number of seconds N in the last 13 billion years (an approximation of the number of seconds since the universe was formed) can be computed as follows:

$$N = \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} \cdot 13 \times 10^9 \text{ years} = 1.1232 \times 10^{15} \text{ seconds.}$$

Now, we must compute the N^{th} partial sum of the harmonic series:

$$\sum_{n=1}^{N}\frac{1}{n} \approx 35.2322,$$

which was obtained using WolframAlpha to compute the sum.

Problem 5. This problem is about error estimation. Please follow the boxed notes about remainder bounds on Page 597 of the text.

(a) **[5 points]** Estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ to within 0.001 of its exact value.

Solution. First, we compute the n-value needed in order to obtain an error less than 0.001. We first verify that $f(x) := \frac{1}{x^4}$ is, indeed, a continuous function of x with only positive terms that decrease as $x \to \infty$, and that they define the terms of the p-series $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^4}$. This particular p-series has $p = 4 > 1$, and so it converges by Example 3 of 10.3. Since these conditions are all met, we may apply the integral test to determine that the remainder $R_n := S - s_n$ (partial sums of the series are s_n and the sum of the series is called S) is bounded by the inequalities

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x.$$

We begin our computation of the proper n-value by computing the integral

$$\int_{k}^{\infty} f(x)\,\mathrm{d}x = \int_{k}^{\infty} \frac{1}{x^4}\,\mathrm{d}x = \lim_{b\to\infty} \int_{k}^{b} \frac{1}{x^4}\,\mathrm{d}x = \lim_{b\to\infty} \left[\frac{-1}{3x^3}\right]_k^b = \lim_{b\to\infty} \frac{-1}{3b^3} + \frac{1}{3k^3} = \frac{1}{3k^3}.$$

This implies that the inequality determining the bounds for R_n may be rewritten:

$$\frac{1}{3(n+1)^3} \leq R_n \leq \frac{1}{3n^3},$$

and so when we seek an error less than $0.001 = 1 \times 10^{-3}$, we seek $n \in \mathbb{N}$ such that

$$\frac{1}{3n^3} \leq 1 \times 10^{-3} \iff n^3 \geq \frac{1}{3} \times 10^3 \iff n \geq \sqrt{3}\frac{1}{3} \times 10^{-3},$$

which is satisfied when $n \geq 7$. We therefore find that

$$\frac{1}{3(8)^3} \leq R_7 \leq \frac{1}{3(7)^3} \iff \frac{1}{1536} \leq R_7 \leq \frac{1}{1029}.$$

We replace R_7 by its definition $R_7 := S - s_7$, computing $s_7 \approx 1.08154$ with WolframAlpha, to obtain

$$\frac{1}{1536} \leq S - s_7 \leq \frac{1}{1029} \iff \frac{1}{1536} + 1.08154 \leq S \leq \frac{1029}{+}1.08154 \iff 1.08605 \leq S \leq 1.08637,$$

and we may estimate S to be the midpoint of the interval $(1.08605, 1.08637)$; that is,

$$S \approx \frac{1.08605 + 1.08627}{2} \approx 1.08616.$$

(b) **[5 points]** How many terms of the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$ should be used to estimate its value with error at most 0.000001 (*i.e.*, at most 1×10^{-6})?

Solution. Let $f(x) := \frac{1}{x^{1.0001}}$, which we observe is continuous, positive, and decreasing for all $x > 0$, and defines the terms of the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$, which as a p-series with $p = 1.0001 > 1$, converges. So we may apply the integral test to see that the remainder R_n is bounded:

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x.$$

We begin our computation of the proper n-value by computing the integral

$$\int_k^{\infty} f(x)\,\mathrm{d}x = \int_k^{\infty} \frac{1}{x^{1.0001}}\,\mathrm{d}x = \lim_{b\to\infty} \int_k^{b} \frac{1}{x^{1.0001}}\,\mathrm{d}x = \lim_{b\to\infty} \left[\frac{-1}{0.0001x^{0.0001}}\right]_k^b = \lim_{b\to\infty} \left[\frac{-10^4}{x^{0.0001}}\right] =$$
$$= \lim_{b\to\infty} \frac{-10^4}{b^{.0001}} + \frac{10^4}{k^{0.0001}} = \frac{10^4}{k^{0.0001}}.$$

This implies that the inequality determining the bounds for R_n may be rewritten:

$$\frac{10^4}{(n+1)^{0.0001}} \leq R_n \leq \frac{10^4}{n^{0.0001}},$$

and so when we seek an error less than 1×10^{-6}, we seek $n \in \mathbb{N}$ such that

$$\frac{10^4}{n^{0.0001}} \leq 1 \times 10^{-6} \iff n^{0.0001} \geq 10^4 \cdot 10^6 \iff n \geq \left(10^{10}\right)^{10^4} = 10^{10\cdot 10^4} = 10^{10^5}.$$

This is a very, very high number—it is 1 with 100,000 zeros after it. Of course, we should have expected this, since the p-series with $p = 1.0001$ is *just* on the convergent side (remember, $p = 1$ gives the harmonic series, which diverges).

Problem 6. This problem is about the comparison test and the limit comparison test. Please follow Section 10.4 of the text.

(a) [**5 points**] If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} \frac{a_n}{n}$? Explain your answer thoroughly!

> **Solution.** Yes; we know that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ must converge as well, but the limit comparison test. We choose to compare to the series $\sum a_n$:
>
> $$\lim_{n\to\infty} \frac{\frac{a_n}{n}}{a_n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$
>
> so case (b) of the limit comparison test applies, and since we were told that $\sum a_n$ converges, then we know that $\sum \frac{a_n}{n}$ must converge as well.

(b) [**5 points**] Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative terms, then $\sum_{n=1}^{\infty} a_n^2$ converges.

> **Solution.** Again, we use the limit comparison test, comparing the series to a_n:
>
> $$\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n = 0,$$
>
> which we know by the contrapositive of the n^{th} term test from Section 10.2. So, case (b) of the limit comparison test applies again, and since we were told that $\sum a_n$ converges, then we know that $\sum \frac{a_n}{n}$ must converge as well.