Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due Tuesday, July 28, 2015, 07:00 p.m. EDT

Week 3: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it.

Reading

Please read Sections 10.7, 10.8, and 10.9 in time for Tuesday's lecture, and Section 10.10 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 10, "Questions to Guide Your Review", p. 647, Problems 22–31

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 10.7, Problems 1–13 odd; 29; 37–43 odd; 49; 53; 53
- Section 10.8, Problems 1–29 odd
- Section 10.9, Problems 1–19 odd; 35–39 odd
- Section 10.10, Problems 1–7 odd; 13–33 odd; 43–49 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Name: ____________________

WPI Username: ____________________

Week 3: Homework Problems

Due date: Tuesday, July 28, 2015, 7:00 p.m. EDT. Please upload a single `.pdf` document to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly.

III) **Show your work**. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. **[10 points]** Prove Nicole Oresme's Theorem:

$$1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots + \frac{n}{2^{n-1}} + \cdots = 4,$$

by differentiating both sides of the equation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

and substituting in an appropriate value of x. In order to get full credit, make sure that you state the reason(s) why you are able to differentiate the series termwise.

Solution. By the definition of geometric series convergence, we know that for $|x| < 1$, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

and so we differentiate both sides of this equation. Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{1-x}\right] = \dfrac{1}{(1-x)^2}$, and when $|x| < 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}[x^n] = \sum_{n=0}^{\infty} nx^{n-1},$$

where we were able to take the derivative of the series termwise because for the values of x we specified ($|x| < 1$), the geometric series is a convergent power series (see Theorem 21 of Section 10.7). Then

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2},$$

and if we substitute the x-value $x = \frac{1}{2}$, we obtain

$$\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4;$$

that is,

$$0 + 1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots + \frac{n}{2^n - 1} + \cdots = 4.$$

Problem 2. The Cauchy condensation test says: Suppose that $\{a_n\}$ is a nonincreasing sequence (*i.e.*, $a_{n+1} \leq a_n$ for all n) with positive terms that converges to 0 as a sequence. Then $\sum a_n$ converges if and only if $\sum 2^n \cdot a_{2^n}$ converges. For example, $\{\frac{1}{n}\}$ is positive and nonincreasing and converges to zero, so we can say that because $\sum 2^n \cdot \frac{1}{2^n} = \sum 1$ diverges, $\sum \frac{1}{n}$ diverges as well. You have the tools to prove why this test works (see Section 10.3), but you do not have to prove this.

(a) [**3 points**] Please use the Cauchy condensation test to show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. Remember to show that the conditions for using the test are satisfied (*i.e.*, that $\{\frac{1}{n \ln n}\}$ is a nonincreasing sequence of positive terms that converges to 0).

Solution. To use the Cauchy condensation test here, we examine the behavior of

$$\sum_{n=2}^{\infty} 2^n \cdot \frac{1}{2^n \ln(2^n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(2^n)} = \sum_{n=2}^{\infty} \frac{1}{n \ln(2)} = \frac{1}{\ln(2)} \sum_{n=2}^{\infty} \frac{1}{n},$$

and by the corollary of Theorem 8 on Page 589 of the text, we see that every nonzero constant multiple of a divergent series—**like the harmonic series, which diverges**—also diverges. So this series diverges, and by the Cauchy condensation test, so does the original series $\sum \frac{1}{n \ln n}$.

(b) [**7 points**] Find the radius of convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$. Get the information you need about the coefficients $\frac{1}{n \ln n}$ from the previous exercise.

Solution. Note that for all $n \in \mathbb{N}$ with $n \geq 2$, $\dfrac{x^n}{n \ln n} \leq x^n$, and note that for $|x| < 1$, we have $\sum_{n=2}^{\infty} x^n$ is a convergent geometric series. Therefore, by the limit comparison test of Section 10.4, we should have that $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$ converges as well, for $|x| < 1$. However, at $x = 1$ we have the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$, which from the previous exercise we know diverges. So the radius of convergence of the series is $R = 1$.

Note that at $x = -1$, we have the sum $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$, which by the Alternating Series Test does converge, but conditionally (you do not need to know this, since it is in Section 10.6, which we did not cover). So the series converges on the half-open interval $[-1, 1)$.

Problem 3. Find the MacLaurin series for...

(a) [**3 points**] $f(x) := \sin(3x)$.

Solution. Since it was proven in Example 2 of Section 10.9 that the MacLaurin series for $\sin(x)$ converges to $\sin(x)$ for all $x \in \mathbb{R}$, we may use the theorems from Section 10.7 about convergent power series; namely, the Substitution Theorem (Theorem 20 of 10.7). We conclude that

$$\sin(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!},$$

which is just $3x$ substituted into the MacLaurin series for $\sin(x)$.

(b) [**3 points**] $f(x) := xe^x$.

Solution. Since it was proven in Example 1 of Section 10.9 that the MacLaurin series for e^x converges to e^x for all $x \in \mathbb{R}$, we may use the theorems from Section 10.7 about convergent power series; namely, the Multiplication theorem (Theorem 19 of 10.7). We conclude that

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}.$$

(c) **[4 points]** $f(x) := \sinh x = \dfrac{e^x - e^{-x}}{2}$.

Solution. We first use the substitution theorem to obtain the MacLaurin series of e^{-x}:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

We then use fact that the sum of two convergent series may be taken termwise (Theorem 8 of 10.2), along with the constant multiple rule (also Theorem 8 of 10.2) to combine the two series:

$$\sinh(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right] = \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n(1+(-1)^n}{n!},$$

and the n^{th} term of this series is 0 when n is odd, but $x^n/n!$ when n is odd. Therefore, the series may be rewritten

$$\sinh(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Problem 4. This problem is about estimating the error in a Taylor polynomial approximation. Please refer to Section 10.9 of the text.

(a) **[5 points]** Estimate the error if $P_3(x) = x - \dfrac{x^3}{6}$ is used to estimate the value of $\sin(x)$ at $x = 0.1$.

Solution. We apply the definition of the remainder:

$$R_3 = \frac{\cos(c)x^4}{4!} = \frac{\cos(c)0.1^4}{4!}$$

for some value c between 0 and $x = 0.1$. When $c \in [0, 0.1]$, $\cos(c) \in [0.995, 1]$ (where the lower limit is approximate), and so we have

$$R_3 \in \left[\frac{0.995}{10^4 \cdot 4!}, \frac{1}{10^4 \cdot 4!}\right] \approx \left[4.14 \times 10^{-6}, 4.16 \times 10^{-6}\right].$$

(b) **[5 points]** Estimate the error if $P_4(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$ is used to estimate the value of e^x at $x = 0.5$.

Solution. We apply the definition of the remainder:

$$R_4 = \frac{e^c x^5}{5!} = \frac{e^c(0.5)^5}{5!} = \frac{e^c}{2^5 \cdot 5!}$$

for some value c between 0 and $x = 0.5$. When $c \in [0, 0.5]$, $e^c \in [1, 1.649]$ (where the upper limit is approximate), and so we have

$$R_4 \in \left[\frac{1}{2^5 \cdot 5!}, \frac{1.649}{2^5 \cdot 5!}\right] \approx \left[2.604 \times 10^{-4}, 4.294 \times 10^{-4}\right].$$

Problem 5. This problem is about Taylor series for even and odd functions.

(a) **[5 points]** Suppose that f is an *even function* of x (*i.e.*, for all x, $f(-x) = f(x)$), whose Maclaurin series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-R, R)$ about $x = 0$.

Prove that $a_1 = a_3 = a_5 = \cdots = 0$; that is, the MacLaurin series for f contains only even powers of x.

Solution. Consider that for an *odd* function $f(x)$, it must be the case that $f(0) = 0$ (indeed, $f(0) = -f(-0) = -f(0)$ implies this). Also consider that the derivative of an odd function is an even function (indeed, the chain rule implies this) and vice-versa. Therefore, for an even function $f(x)$, we will have that $f'(x)$ is odd, and so $f'(0) = 0$, which implies $a_1 = 0$. In fact, for any odd n, $f^{(n)}(0) = 0$ implies that $a_n = 0$, which proves that the MacLaurin series contains only even powers of x.

(b) [**5 points**] Suppose that f is an *odd function* of x (*i.e.*, for all x, $f(-x) = -f(x)$), whose Maclaurin series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-R, R)$ about $x = 0$.

Prove that $a_0 = a_2 = a_4 = \cdots = 0$; that is, the MacLaurin series for f contains only odd powers of x.

Solution. The proof here is similar; for an odd function $f(x)$, we have $f(0) = 0$, and so $a_0 = 0$, and for every even n, $f^{(n)}(0) = 0$ which implies that $a_n = 0$, and so the MacLaurin series for f contains only odd powers of x.

Problem 6. [**10 points**] Find the first four nonzero terms in the MacLaurin series for the function $f(x) := \cos^2(x) \cdot \sin x$. Plot $f(x)$ on the same axes as the polynomial approximation for $x \in (-5, 5)$ (you may use a computer or graphing calculator to do this; remember to include the correct axis labels, graph labels, and number labels on your axis ticks).

Solution. There were two ways to do this—the first one uses the fact that cos and sin both have MacLaurin series that converge for all x, and so the multiplication theorem from Section 10.7 could be applied twice over (once to obtain the series for $\cos^2$ and once more for $\cos^2 \sin$). We could also compute the coefficients directly, which is what I show now. The differentiation gets a little hairy, so I use WolframAlpha for this (that's okay on your homework assignments, for the portions of the problem that are not being evaluated as new course material).

The first thing to observe about f is that it is an odd function, and so all even Taylor coefficients will be 0; we therefore do not even evaluate $f(0)$ (we know that it is zero anyway), and go straight to the odd-numbered derivatives:

$$\begin{aligned}
f'(x) &= \frac{1}{4}\left(\cos(x) + 3\cos(3x)\right) \implies f'(0) = \frac{1}{4}(1+3) = 1 \\
f'''(x) &= 20\sin^2 x \cos x - 7\cos^3 x \implies f'''(0) = -7 \\
f^{(5)}(x) &= \frac{1}{4}(\cos x + 243\cos(3x)) \implies f^{(5)}(x) = \frac{1+243}{4} = 61 \\
f^{(7)}(x) &= \frac{1}{4}(-\cos x - 2187\cos(3x)) \implies f^{(7)}(x) = \frac{-1-2187}{4} = -547.
\end{aligned}$$

Therefore, the first four nonzero terms in the MacLaurin series are:

$$x - \frac{7}{3!}x^3 + \frac{61}{5!}x^5 - \frac{547}{7!}x^7.$$

When we plot $g(x) := x - \frac{7}{3!}x^3 + \frac{61}{5!}x^5 - \frac{547}{7!}x^7$ on the same axes as $f(x)$ for $x \in (-5, 5)$, we obtain