Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due Monday, August 10, 2015, 11:59 p.m. EDT

Week 5: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it.

Reading

Please read Sections 11.3, 11.4, and 11.5 in time for Tuesday's lecture, and 12.1, 12.3, 12.3, and 12.4 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 11, "Questions to Guide Your Review", p. 699, Problems 9–13
- Chapter 12, "Questions to Guide Your Review", p. 745, Problems 1–10

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 11.3, Problems 1–23 odd; 41–61 odd
- Section 11.4, Problems 1–31 odd
- Section 11.5, Problems 1–29 odd
- Section 12.1, Problems 1–27 odd; 35–49 odd
- Section 12.2, Problems 1–39 odd
- Section 12.3, Problems 1–25 odd
- Section 12.4, Problems 1–27 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Name: ______________________

WPI Username: ______________________

Week 5: Homework Problems

Due date: Monday, August 10, 2015, 11:59 p.m. EDT. Please upload a single `.pdf` document to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly.

III) **Show your work**. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. Replace the Cartesian equations with equivalent polar equations. Please use Section 11.3 as reference.

(a) [**2 points**] $x = a$, where a is a constant. (The Cartesian equation is a vertical line.)

Solution. We use $x = r\cos\theta$ to obtain $r\cos\theta = a$, or equivalently, $r = \frac{a}{\cos\theta}$, for $\theta \in (-\pi/2, \pi/2)$.

(b) [**2 points**] $y = b$, where b is a constant. (The Cartesian equation is a horizontal line.)

Solution. We use $y = r\sin\theta$ to obtain $r\sin\theta = b$, or equivalently, $r = \frac{b}{\sin\theta}$, for $\theta \in (0, \pi)$.

(c) [**3 points**] $x^2 + xy + y^2 = 1$.

Solution. We use $x = r\cos\theta$ and $y = r\sin\theta$ to determine that $x^2 + y^2 = r^2$, and therefore,

$$r^2 + r^2\sin\theta\cos\theta = 1 \iff r^2(1 + \sin\theta\cos\theta) = 1 \iff r^2 = \frac{1}{1 + \sin\theta\cos\theta},$$

where we note that $1 + \sin\theta\cos\theta \neq 0$ for any value of θ.

(d) [**3 points**] $(x + 2)^2 + (y - 5)^2 = 16$.

Solution. We first rewrite the equation to obtain

$$x^2 + 4x + 4 + y^2 - 10y + 25 = 16 \iff x^2 + y^2 + 4x - 10y = -13.$$

Then we use $x = r\cos\theta$ and $y = r\sin\theta$ to determine that $x^2 + y^2 = r^2$, and therefore,

$$r^2 + r(4\cos\theta - 10\sin\theta) + 13 = 0,$$

or

$$\begin{aligned} r &= \frac{-(4\cos\theta - 10\sin\theta) \pm \sqrt{(4\cos\theta - 10\sin\theta)^2 - 4(1)(13)}}{2(1)} \\ &= 5\sin\theta - 2\cos\theta \pm \sqrt{4\cos^2\theta - 10\cos\theta\sin\theta + 25\sin^2\theta - 13}, \end{aligned}$$

provided that the discriminant is nonnegative.

Problem 2. Graph the following *limaçons*, without using graphing software or computational aids. Please see Section 11.4.

Solution. For this problem, I was checking that you looked for possible symmetries of the graph; for the slope of the graph at the origin; and for possible periodicity of the graph, along with creating a table of r-values for an appropriate range of θ values. Many of you found it useful to use parametric graph paper for this problem.

Note that $\cos\theta = \cos(-\theta)$ for all values of θ (*i.e.*, cos is an even function), and so each of the limaçons below exhibits symmetry about the x-axis.

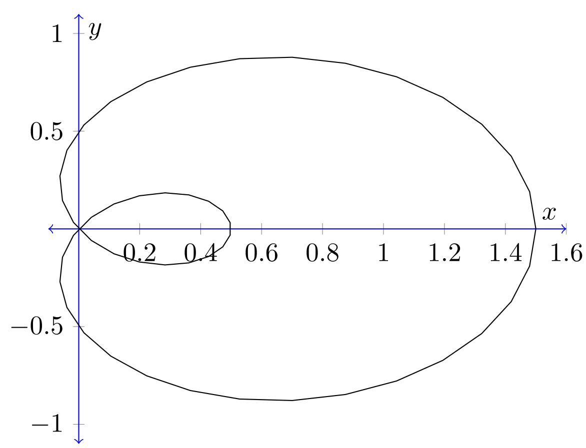
Only the first two limaçons have graphs that pass through the origin; for the first limaçon, this occurs when $\cos\theta = -1/2$, that is, when $\theta = 2\pi/3$, and at this value, the slope of the graph is equal to $\tan(2\pi/3) = -1$. For the second limaçon, the graph passes through the origin when $\cos\theta = 1$, that is, when $\theta = 0$, and at this value, the slope of the graph is equal to $\tan(0) = 0$.

Finally, each of the functions is 2π periodic, since each is the sum of a constant and $\cos\theta$. The observations about symmetry, periodicity, and slope at the origin will all help us with the graphs of these figures.

(a) [2 points] $r = \frac{1}{2} + \cos\theta$ (this limaçon has an inner loop).

Solution. Our table of values may look as follows (we take $\theta \in [0, \pi]$ and not $[0, 2\pi]$ because of symmetry:

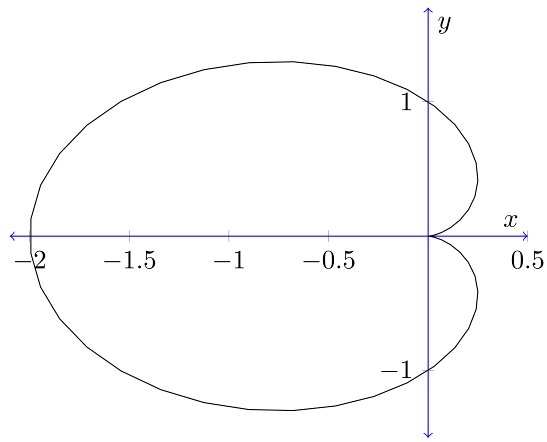
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$r = \frac{1}{2} + \cos\theta$	$\frac{3}{2}$	$\frac{1+\sqrt{3}}{2}$	$\frac{1+\sqrt{2}}{2}$	1	$\frac{1}{2}$	0	$\frac{1-\sqrt{2}}{2}$	$\frac{1-\sqrt{3}}{2}$	$-\frac{1}{2}$
Approximate r	$\frac{3}{2}$	1.366	1.207	1	$\frac{1}{2}$	0	-0.207	-0.366	$-\frac{1}{2}$



(b) **[2 points]** $r = 1 - \cos\theta$ (this limaçon looks like a cardioid).

Solution. Our table of values may look as follows (we take $\theta \in [0, \pi]$ and not $[0, 2\pi]$ because of symmetry:

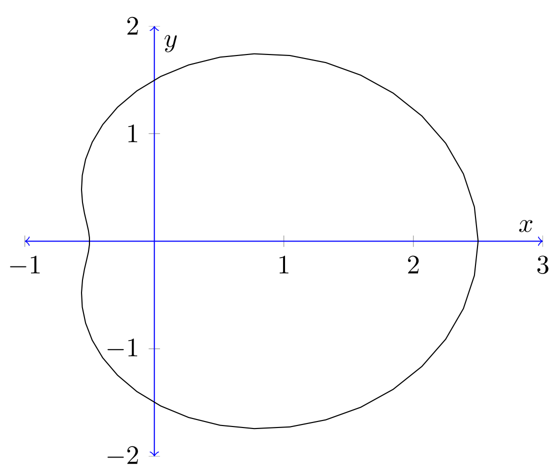
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$r = 1 - \cos\theta$	0	$\frac{2-\sqrt{3}}{2}$	$\frac{2-\sqrt{2}}{2}$	$-\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{2+\sqrt{2}}{2}$	$\frac{2+\sqrt{3}}{2}$	2
Approximate r	0	0.134	0.293	$-\frac{1}{2}$	1	$\frac{3}{2}$	1.707	1.866	2



(c) **[3 points]** $r = \frac{3}{2} + \cos\theta$ (this limaçon has a dimple instead of a loop).

Solution. Our table of values may look as follows (we take $\theta \in [0, \pi]$ and not $[0, 2\pi]$ because of symmetry:

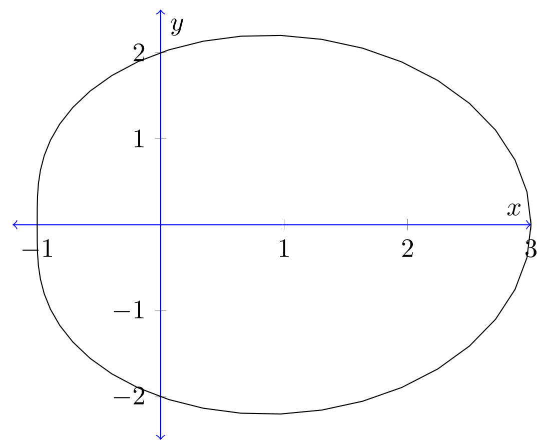
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$r = \frac{3}{2} + \cos\theta$	$\frac{5}{2}$	$\frac{3+\sqrt{3}}{2}$	$\frac{3+\sqrt{2}}{2}$	2	$\frac{3}{2}$	1	$\frac{3-\sqrt{2}}{2}$	$\frac{3-\sqrt{3}}{2}$	$\frac{1}{2}$
Approximate r	$\frac{5}{2}$	2.366	2.207	2	$\frac{3}{2}$	1	0.792	0.633	$\frac{1}{2}$



(d) [3 points] $r = 2 + \cos\theta$ (this limaçon looks like an oval).

Solution. Our table of values may look as follows (we take $\theta \in [0, \pi]$ and not $[0, 2\pi]$ because of symmetry:

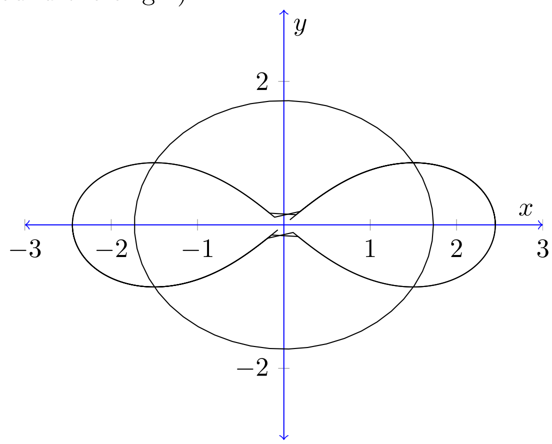
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$r = 2 + \cos\theta$	3	$\frac{4+\sqrt{3}}{2}$	$\frac{4+\sqrt{2}}{2}$	$\frac{5}{2}$	2	$\frac{3}{2}$	$\frac{4-\sqrt{2}}{2}$	$\frac{4-\sqrt{3}}{2}$	1
Approximate r	3	2.866	2.707	$\frac{5}{2}$	2	$\frac{3}{2}$	1.292	1.134	1



Problem 3. These problems are about finding the areas inside and between polar curves. Please see Section 11.5.

(a) **[5 points]** Find the area inside the leminiscate $r^2 = 6\cos(2\theta)$ and outside the circle $r = \sqrt{3}$.

Solution. The curve defining the leminiscate and the curve defining the circle are both smooth, and the leminiscate is traced out exactly once as θ ranges from $-\pi/6$ to $\pi/6$ (see image below, but please ignore the plotting errors around the origin).



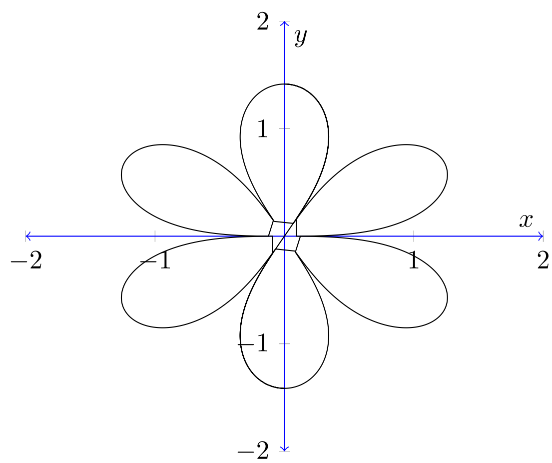
Keeping in mind that there are two lobes of the leminiscate to account for, we use the formula

$$A = 2 \cdot \frac{1}{2} \int_{-\pi/6}^{\pi/6} r_2^2 - r_1^2 \, \mathrm{d}\theta = 2 \int_0^{\pi/6} 6\cos(2\theta) - 3 \, \mathrm{d}\theta = 6 \left[\sin(2\theta) - \theta\right] \Big|_0^{\pi/6} = 6 \left[\sin(\pi/3) - \frac{\pi}{6}\right] = 3\sqrt{3} - \pi,$$

with the second equality arising from the fact that the integrand is even and the integral is taken on an interval symmetric about the origin.

(b) **[5 points]** Find the area inside the six-leaved rose $r^2 = 2\sin(3\theta)$.

The curve defining the rose is smooth, and one half petal of the rose is traced out exactly once as θ ranges from 0 to $\pi/6$ (see image below, but please ignore the plotting errors around the origin).



Keeping in mind that there are twelve half-petals of the rose to account for, we use the formula

$$A = 12 \cdot \frac{1}{2} \int_0^{\pi/6} r^2 \, \mathrm{d}\theta = 6 \int_0^{\pi/6} 2\sin(3\theta) \, \mathrm{d}\theta = 12 \left[-\frac{1}{3}\cos(3\theta)\right] \Big|_0^{\pi/6} = -4\left[\cos(\pi/2) - \cos(0)\right] = 4.$$

Problem 4. [10 points] Vectors are drawn from the center of a regular[2] n-sided polygon in the plane to the vertices of the polygon. Show that the sum of the vectors is zero. [Hint: What happens to the sum of the vectors if you rotate the polygon about its center?]. **You must give a well-structured and coherent logical argument, using sentences, to justify your answer. Your answer should not just be a string of calculations or a mess of arrows and diagrams. Also, an example does not constitute a general proof: do not just show this for a 3-sided or 4-sided polygon!**

Solution. Follow the hints in the e-mail. First, notice that when we rotate two vectors $\vec{u}$ and $\vec{v}$ by the same angle θ (we'll write the rotated vectors as $\vec{u}'$ and $\vec{v}'$ respectively), then the sum of the rotated vectors $\vec{u}' + \vec{v}'$ is equal to the rotated sum of the original vectors $(\vec{u} + \vec{v})'$. This is simple to prove using the geometric definition of vector sums (see the Parallelogram Law on Page 711), but those of you who've already worked with matrix operations can also think of the rotating "function" $f(\vec{v}) = \vec{v}'$ as multiplication by a rotation matrix, in which case the proof is also simple: just an application of the linearity of matrix multiplication. This result may be extended to any sum of finitely many vectors (that is, $\vec{u}_1' + \vec{u}_2' + \cdots + \vec{u}_n' = (\vec{u}_1 + \vec{u}_2 + \cdots + \vec{u}_n)'$, for any natural number n).

Second, unless the angle θ is a multiple of 2π, then the only vector for which $\vec{v} = \vec{v}'$ is the zero vector $\vec{0}$.

Third, suppose that we have a regular n-sided polygon in the plane; when this polygon is rotated through the angle $\theta = \frac{2\pi}{n}$, then the resulting shape is an identical polygon, where for an original sequential labelling of the vertices v_1 through v_n, we have that $\vec{v}_1' = \vec{v}_2$, $\vec{v}_2' = \vec{v}_3$, and so forth, until $\vec{v}_n' = \vec{v}_1$. Therefore, we have that

$$\vec{S} := \sum_{i=1}^{n} \vec{v}_i = \sum_{i=1}^{n} \vec{v}_i' = \left(\sum_{i=1}^{n} \vec{v}_i \right)' = \vec{S}'.$$

But we had already shown that when θ is not a multiple of 2π (and here, it is not), then $\vec{v} = \vec{v}'$ implies $\vec{v} = \vec{0}$. Therefore, $\vec{S} = \vec{0}$, just what we wanted to prove.

Problem 5. [10 points] This problem is about the famous Cauchy-Schwartz inequality. Here, we will prove this inequality in $\mathbb{R}^3$, but it holds in any vector space[3].

(a) [5 points] Since $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$, show that the inequality $|\vec{u} \cdot \vec{v}| \leq |\vec{u}|\,|\vec{v}|$ holds for any vectors $\vec{u}$ and $\vec{v}$.

Solution. $|\vec{u} \cdot \vec{v}| = ||\vec{u}|\,|vv| \cos\theta| = |\vec{u}|\,|\vec{v}|\,|\cos\theta|$, and $|\cos\theta| \leq 1$ for all θ; therefore, $|\vec{u} \cdot \vec{v}| \leq |\vec{u}|\,|vv|$.

(b) [5 points] Under what circumstances, if any, does $|\vec{u} \cdot \vec{v}|$ equal $|\vec{u}|\,|\vec{v}|$? You must justify your answer.

Solution. We will have $|\vec{u} \cdot \vec{v}| = |\vec{u}|\,|\vec{v}|$ only when $|\cos\theta| = 1$; that is, when $\theta = n\pi$, for $n \in \mathbb{Z}$. This means that the two vectors would be scalar multiples of one another (when n is odd, that corresponds to a negative scalar, and when n is even or zero, that corresponds to a positive or zero scalar, respectively).

Problem 6. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

(a) [2 points] $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

Solution. Always true:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}.$$

(b) [2 points] $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$

Solution. Never true (unless one or both of $\vec{u}$ and $\vec{v}$ is zero, or $\vec{u}$ and $\vec{v}$ are parallel), since $\vec{u} \times \vec{v} = |\vec{u}|\,|\vec{v}| \sin\theta \hat{n}$, and when the order of the vectors in the product is switched, then $\hat{n}$, which by definition is computed according to the right-hand rule, points in exactly the opposite direction, making the cross product switch sign. That is, $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

[2] A regular polygon is one that has all angles equal in measure, and all sides equal in length. Like an equilateral triangle (a regular 3-sided polygon), or a square (a regular 4-sided polygon).

[3] Read more about the inequality here: `https://en.wikipedia.org/wiki/Cauchy\%E2\%80\%93Schwarz_inequality` .

(c) **[2 points]** $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$ for any number c

Solution. Always true; can prove either using the determinant definition of the cross product, or the geometric definition, where $|c\vec{u}| = |c|\,|\vec{u}|$ and $|c\vec{v}| = |c|\,|\vec{v}|$, observing that if c is negative, then multiplying one of the vectors $\vec{u}$ or $\vec{v}$ by c does change their direction, and therefore the cross product itself will also change direction accordingly.

(d) **[2 points]** $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

Solution. Always true:

$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = |\vec{u}|^2 .$$

(e) **[2 points]** $(\vec{u} \times \vec{v}) \cdot \vec{u} = \vec{v} \cdot (\vec{u} \times \vec{v})$

Solution. Always true. Observe that by the definition of the cross product, $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$; that is, $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ and $(vu \times \vec{v}) \cdot \vec{v} = 0$.