Lecture 5:
Lesson and Activity Packet

MATH 232: Introduction to Statistics

September 23, 2016

Homework and Announcements

- Don't leave class today without submitting Homework 3.
- Homework 4 due Monday in class.

Recap

Last time, we discussed:

- Fractiles (measures of position)
  - Median (this is not new)
  - Quartiles
  - Deciles
  - Percentiles
- Interquartile range
- Box-and-Whisker Plot

Questions?

If not, then today, we begin **Chapter 4: Probability.**

To preface our study of probability, we should speak in a standard way about what is generally possible in the world; we can't predict the outcome of a soccer game if we don't know who's playing, for example, just as we can't predict an election without knowing who's running.
We start with some terminology.

Definition 1

- *An* experiment *is any operation or procedure whose outcomes cannot be predicted with certainty.*
- *The set of all possible* outcomes *for the experiment is called the* sample space *of the experiment. The sample space is a set (i.e., an unordered list), and is usually given the name S.*

Example 1
The single toss of a coin is an ***experiment****, because its* ***outcomes*** *(heads, tails) cannot be predicted with certainty. The* ***sample space*** *of the coin-toss experiment can be written as* $S := \{H, T\}$.

Example 2
The single toss of a die is an ***experiment*** *with one of six* ***outcomes****. Here, the* ***sample space*** *is* $S = \{1, 2, 3, 4, 5, 6\}$.

Group Exercise 1 (*1 minute*)
How many outcomes are in the sample space corresponding to the experiment of selecting one card from a standard deck?

$S = \{1, 2, 3, \ldots, 51, 52\}$ or $S = \{A\heartsuit, 2\heartsuit, 3\heartsuit, \ldots, Q\heartsuit, K\heartsuit,$
$A\diamondsuit, 2\diamondsuit, 3\diamondsuit, \ldots, Q\diamondsuit, K\diamondsuit,$
$\ldots, \ldots, Q\clubsuit, K\clubsuit\}$

and in either case, $|S| = 52$.

vertical bars around a set's name denote cardinality of the set - i.e., how many elements it contains

Sometimes events consist of more than one trial or component: each **outcome** can be decomposed into one or more outcomes.

Example 3

When classifying people by their blood type, usually the antigen type (A, B, AB, or O) is given along with the Rhesus factor (+,-). Your instructor's blood type is O+, for example.

Group Exercise 2

List all of the possible blood types. (This is the sample space.) How many blood types are there?

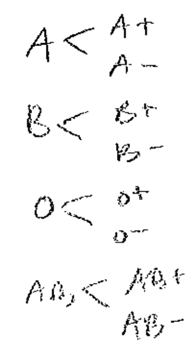
$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$

$|S| = 8$

A *tree diagram* is a visual representation of the number of possible outcomes of multi-step experiments.

Example 4

For the blood type data, the tree diagram is:



Theorem 1 (***The basic principle of counting***)

Suppose that two experiments are to be performed. If experiment 1 can result in any one of m possible outcomes and if for each result of experiment 1, experiment 2 can result in n possible outcomes, then taken together there are mn possible outcomes of the two experiments.

Example 5

There were four ways to choose the antigen type, and two ways to choose the Rhesus factor; so there are $4 \cdot 2 = 8$ possible blood types.

Example 6

If an ice cream shop has 20 flavors to choose from with 8 choices of toppings, then there are a total of $20 \cdot 8 = 160$ different combinations you can choose from.

Theorem 2 (*The generalized principle of counting*)

If k experiments are to be performed, and the first one may result in any of n_1 possible outcomes; the second may result in n_2 possible outcomes, and so on, then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_k$ possible outcomes of the k experiments.

Example 7

How many 7-place license plates are possible if the first three places are letters, and the final four places are numbers?

$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 26^3 \cdot 10^4 = 175{,}760{,}000$

Example 8

How many 5-card poker hands are there?

$\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48} = 311{,}875{,}200$

Example 9

How many license plates would be possible if repetition among letters or numbers were prohibited?

$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78{,}624{,}000$

Example 10

How many ways are there to arrange 10 objects from left to right on a shelf?

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10! \sim 3{,}628{,}800$

Definition 2

- *An event is any collection of results or outcomes of a procedure. An event is a set, and it is a subset of the sample space (i.e., each member of an event must have appeared in the sample space first). Events are usually given the name E. We therefore write $E \subseteq S$, pronounced "E is a subset of S".*
- *A simple event is a set with exactly one outcome.*
- *A compound event contains more than one outcome.*

Example 11

- *If an experiment/procedure consists of flipping two coins, then the* ***sample space*** *contains the following four points: $S_1 := \{(H,H),(H,T),(T,H),(T,T)\}$.*
- *A* ***simple event*** *could be $E = (H,H)$ (both coins come up heads).*
- *A* ***compound event*** *might be $E = \{(H,H),(H,T)\}$ (the event that a head appears on the first coin).*

Example 12

- *If an experiment consists of tossing two dice, then the* ***sample space*** *consists of the 36 ordered pairs in the set $S_2 := \{(i,j),$ where i,j are among $1,2,3,4,5,6\}$.*
- *A* ***compound event*** *might be $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ (the event that the sum of the dice is 7).*

It is often important to know how many outcomes a sample space or an event contains. We already have the generalized rule of counting for simple cases; we will now learn about **permutations** and **combinations**.

Example 13

Recall the example where you placed 7 distinct items on a shelf from left to right: we used the generalized rule of counting to figure out that there were $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ *ways to do it.*

Definition 3

A *permutation* is a way of arranging objects in an ordered list.

Group Exercise 3 (*30 seconds*)

How many permutations are there for a set of 7 distinct items?

Same as shelf problem : 5,040

Group Exercise 4 (*5 minutes*)

How many permutations are there for a set of n distinct items? Does any of your groupmates know the "factorial" notation? If so, express the number of permutations using that notation.

$n(n-1)(n-2)\cdots(3)(2)(1) =: n!$

Group Exercise 5 (*1 minute*)

How many different batting orders are there for a baseball team consisting of 9 players?

$9! = 362,880$

One important variation on permutations is the problem of selecting r items from a set of n distinct items. We can figure this out using the rule of counting as well.

Example 14

How many ways are there to select two items from a set of 26?

$\underline{26} \cdot \underline{25} = 650$

choices for 1st item

choices for 2nd item

Group Exercise 6 (*2 minutes*)

In how many different ways can our class (which we can assume has 26 students) select a president, a vice president, a secretary, and a treasurer, assuming that no single student can hold more than one position?

$\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} = 358,800$

Group Exercise 7 (*Optional*)

How do you use factorial notation to express the total number of ways that r objects can be selected from n distinct objects[a]? This number is typically given the name ${}_nP_r$.

[a]The exercise here is optional, but the concept is not: if you choose not to figure this out as an exercise, you'll still need to listen to the instructor talk about it and write down notes.

$${}_nP_r = \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{\cdots 3 \cdot 2 \cdot 1}}{\cancel{(n-r)(n-r-1)\cdots 3 \cdot 2 \cdot 1}}$$

Test: $${}_{26}P_2 = \frac{26!}{(26-2)!} = \frac{26!}{24!} = \frac{26 \cdot 25 \cdot \cancel{24 \cdot 23 \cdot 22 \cdots 3 \cdot 2 \cdot 1}}{\cancel{24 \cdot 23 \cdot 22 \cdots 3 \cdot 2 \cdot 1}}$$

$= 26 \cdot 25$ ✓

There are several other important variations on the permutation problem. Here are examples of two more variations:

Example 15

I have 10 different books that I want to put on my bookshelf: 4 are mathematics books, 3 are language books, 2 are history books, and 1 is a chemistry book. But I want to keep all books with the same subject together. How many different arrangements are possible?

4! many ways to arrange math books

3! —— " —— language "

2! —— " —— history "

1! —— " —— chem " .

and 4! many ways to arrange M-L-H-C blocks

So, by generalized counting: $(4!)(3!)(2!)(1!)(4!) = 6{,}912$

Example 16

How many different letter arrangements can be formed using the letters PEPPER? [The fact that several of the letters repeat in this problem demonstrates why we kept using the words ***"distinct"*** *and* ***"different"*** *in previous examples!]*

6 letters total - if they were distinct, would be $6! = 720$ many permutations. But the three P's are indistinct, and the two E's are indistinct.

That means that the 720 permutations really come in groups of $(3!)(2!) = (3\cdot 2)(2) = 6\cdot 2 = 12$ per group, where the 12 permutations in each group are identical. For example, if we number the letters: $P_1E_1P_2P_3E_2R$, then one group consists of the 12 arrangements: $P_1P_2E_1P_3E_2R$, $P_1P_2E_2P_3E_1R$, $P_1P_3E_1P_2E_2R$, $P_1P_3E_2P_2E_1R$, $P_2P_1E_1P_3E_2R$, $P_2P_1E_2P_3E_1R$, $P_2P_3E_1P_1E_2R$, $P_2P_3E_2P_1E_1R$, $P_3P_1E_1P_2E_2R$, $P_3P_1E_2P_2E_1R$, $P_3P_2E_1P_1E_2R$, $P_3P_2E_2P_1E_1R$.

Combinations are a special case of the "choosing r objects from n many" problem, where the order of the chosen objects doesn't matter.
Remember the example:

Example 17

In how many different ways can our class (which we can assume has 26 students) select a president, a vice president, a secretary, and a treasurer, assuming that no single student can hold more than one position?

(done before) $26\cdot 25\cdot 24\cdot 23 = 358{,}800$

Consider the similar example:

Example 18

In how many different ways can our class (which we can assume has 26 students) select a council of four students to represent us at a statistics conference?

The crucial difference here is that in the first example, the four positions were distinct. If Hannah is the president, Mykaela is the vice president, Nicole is the treasurer, and Tim is the secretary, then this is a **different** situation than having Mykaela be the president, Hannah the vice president, Nicole the treasurer, and Tim the secretary.

However, in the second example, order does not matter. If the group consists of Tiffany, Kaitlyn, Paul, and Akira, then the group with Kaitlyn, Akira, Paul, and Tiffany is exactly the same.

So, let's answer the question: how many ways can 4 students represent 26?

There are 358,800 ways to choose if order matters; these choices come in groups of $4! = 4\cdot 3\cdot 2 = 24$ councils that have the same members.

So there are $\frac{358{,}800}{24} = 14{,}950$ many distinct groups.

That is, $\frac{26!}{(26-4)!\,4!} = \frac{26!}{22!\cdot 4!} = 14{,}950$.

This means the total # of arrangements of the letters PEPPER is:

$\frac{6!}{(3!)(2!)} = \frac{720}{12} = 60$ possible arrangements

The number of combinations of n many objects taken r at a time is denoted $_nC_r$ or $\binom{n}{r}$, and is given by the formula

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Group Exercise 8 (*2 minutes*)

Our favorite ice cream shop is about to close up for the season, so its offerings are scant. They have vanilla, dark chocolate, maple walnut, coffee, pistachio, and raspberry. If you know you want a cone with three scoops of ice cream (and if it doesn't matter to you which flavor is on top, etc.), then how many possible combinations are there?

6 flavors - want 3 scoops, order doesn't matter.

$$6C3 = \binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!\cdot 3!} = \frac{\cancel{6}\cdot 5\cdot 4\cdot \cancel{3\cdot 2}}{(\cancel{3\cdot 2})(3\cdot 2)} = 5\cdot 4 = 20$$

Group Exercise 9 (*optional; 2 minutes*)

*Think back to the example of choosing 4 students on the council to represent our class of 26 at a statistics meeting. How many ways are there of choosing 22 students who do **not** serve on that council? Think before computing!*

Should be the same as choosing 4 who do serve — so 14,950.

Group Exercise 10 (*optional; 2 minutes*)

Show using the formula that $\binom{n}{r} = \binom{n}{n-r}$.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{and} \quad \binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!(n-n+r)!} \quad \text{distributive law}$$

same

$$= \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n!}{r!\,(n-r)!} \quad \text{commutative law}$$

Recap

- Experiment, outcome, sample space
- Sets and subsets
- Counting
  - Tree diagrams
  - Basic principle of counting
  - Generalized principle of counting
- Events
  - Simple events
  - Compound events
- Permutations
  - Permutations for sets of distinct items
  - Selecting r ordered items from n distinct items
  - Groups of permutations (bookshelf problem)
  - Permutations for sets of non-distinct items ($PEPPER$ problem)
  - $_nP_r$
  - Factorials
- Combinations
  - Selecting r unordered items from n distinct items
  - $_nC_r = \binom{n}{r}$
  - $\binom{n}{r} = \binom{n}{n-r}$