

Shakespeare: 11/12 7p.m.
Church of Social Hall
\$5 Faculty

Lecture 12: Lesson and Activity Packet

MATH 232: Introduction to Statistics

October 31, 2016

Homework and Announcements

- Homework 13 due in class on Friday (find it posted on Canvas tonight)
- Election Prediction Project
 - Deadline for ASA submission was yesterday
 - Deadline for group report, proof of ASA submission, and individual report is tonight at 11:59 p.m.
- Quiz 6 on Wednesday will cover everything through today
- Exam 2 is one week from Wednesday!
 - It will cover everything we do this week.
 - Monday (one week from today) will be a review day.
- Book Review Project
 - Details to be given after the Election Project, **but** you should buy or procure (by legal means...) the book of your choice now, from among the following three choices:
 1. *Moneyball: The Art of Winning an Unfair Game*, by Michael Lewis
 2. *Dataclysm: Love, Sex, Race, and Identity—What Our Online Lives Tell Us about Our Offline Selves*, by Christian Rudder
 3. *How Not to Be Wrong*, by Jordan Ellenberg
 - You must make your choice before Monday, and **use groups on Canvas to sign up for your desired book**

Last time

- Conditional probability
- Independent events
- Multiplication rules

Recap of today

- More independent events and multiplication rules
- Bayes' Theorem (?)

Last time, we had the definition of *independent events*:

Definition 1 (*Independent Events*)

If $P(B) \neq 0$ and if $P(A|B) = P(A)$, we say that event A is independent of event B . That is, event A is independent of event B if the probability of A is not affected by the occurrence or non-occurrence of event B .

Group Exercise 1

The probabilities that it will rain or snow in a given city on Christmas Day, on New Year's Day, or on both days, are $P(C) := 0.60$, $P(N) := 0.60$, and $P(C \cap N) := 0.42$. Check whether N and C are independent.

$$P(N|C) = \frac{P(N \cap C)}{P(C)} = \frac{P(C \cap N)}{P(C)} = \frac{0.42}{0.6} = 0.7 \neq P(N),$$

So the events are not independent.

Group Exercise 2

If $P(A) = 0.80$, $P(C) = 0.95$, and $P(A \cap C) = 0.76$, are events A and C independent?

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.76}{0.8} = 0.95, \quad \cancel{=}$$

$= P(A)$, so yes, A & C

are independent.

Some interesting Multiplication Rules follow from the definition of the conditional probability of E given F :

$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)}.$$

If we multiply both sides of this equation by $P(F)$, we obtain one version of the general multiplication rule:

Theorem 1

General multiplication rule

$$P(E \text{ and } F) = P(F) \cdot P(E|F)$$

But also, the definition of the conditional probability of F given E is:

$$P(F|E) = \frac{P(F \text{ and } E)}{P(E)}.$$

Noticing that for all events A and B , $P(A \text{ and } B) = P(B \text{ and } A)$, the conditional probability of F given E becomes

$$P(F|E) = \frac{P(E \text{ and } F)}{P(E)},$$

and we can again multiply both sides of the equation by $P(E)$ to obtain **another** version of the general multiplication rule:

Theorem 2

General multiplication rule

$$P(E \text{ and } F) = P(E) \cdot P(F|E).$$

Both of these rules hold for **all** events E and F .

Example 1

A jury consists of nine men and three women. If two of the jurors are randomly picked for an interview, what is the probability that they will both be women?

Let A be the event ~~that~~ the first juror is a woman, and B be the event that the second juror is a woman. The probability we seek is $P(A \text{ and } B)$.

Because we are selecting jurors at random, the probability that the first juror picked will be a woman is $P(A) = \underline{\hspace{2cm}}$.

If the first juror is already picked and ends up being a woman, then we have 11 jurors left to choose from, two of whom will be women. This means that $P(B|A) = \underline{\hspace{2cm}}$. [Notice: we didn't need to use the formula for conditional probability at this stage of the problem—just our general knowledge of classical probability. This is okay, and in fact, it's what's intended!]

We then use the multiplication rule to compute $P(A \text{ and } B)$:

$$P(A \text{ and } B) = P(A) \cdot P(B|A) =$$

$$P(A) = \frac{3}{12} = \frac{1}{4} = 0.25 \text{ or } 25\%$$

$$P(B|A) = \frac{3-1}{12-1} = \frac{2}{11}$$

$$\begin{aligned} \text{So } P(A \text{ and } B) &= P(A) \cdot P(B|A) = \frac{1}{4} \cdot \frac{2}{11} = \frac{1 \cdot 2}{2 \cdot 11} = \frac{1}{2 \cdot 11} = \frac{1}{22} \\ &= \frac{3}{12} \cdot \frac{2}{11} \approx 4.5\% \end{aligned}$$

If A and B are independent events, then recall that this means $P(B|A) = P(B)$, and $P(A|B) = P(A)$. For two events that are **already known to be independent**, the general multiplication rule then boils down to:

Theorem 3 (General multiplication rule for independent events)

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = P(A) \cdot P(B).$$

Example 2

What is the probability of getting heads in two flips of a balanced coin?

Since the two events are independent, and the probability of heads is $\frac{1}{2}$ for each flip, the answer is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Example 3

Revisiting the snowing-on-Christmas example: The probabilities that it will rain or snow in a given city on Christmas Day, on New Year's Day, or on both days, are $P(C) := 0.60$, $P(N) := 0.60$, and $P(C \cap N) := 0.42$. Check whether N and C are independent.

$$P(C) \cdot P(N) \stackrel{?}{=} P(C \text{ and } N) \quad \dots \text{ well, } P(C)P(N) = 0.6 \cdot 0.6 = 0.36 \\ \neq P(C \text{ and } N) = 0.42, \\ \text{So no - not indep.}$$

Example 4

If A and B are independent events and $P(A) = 0.20$ and $P(B) = 0.45$, find:

(a) $P(A \text{ and } B) = P(A)P(B) = 0.2 \cdot 0.45 = 0.09$

(b) $P(A|B) = P(A) = 0.2$

(c) $P(B|A) = P(B) = 0.45$

(d) $P(A^C \text{ and } B) = P(A^C)P(B) = (1 - P(A))P(B) = (1 - 0.2)0.45 = 0.8 \cdot 0.45 = 0.36$

(e) $P(A \text{ and } B^C) = P(A)P(B^C) = P(A)(1 - P(B)) = 0.2(1 - 0.45) = 0.2 \cdot 0.55 = 0.11$

(f) $P(B^C|A^C) = P(B^C) = 1 - 0.45 = 0.55$

(g) $P(A^C \text{ and } B^C) = P(A^C)P(B^C) = (1 - P(A))(1 - P(B)) = (1 - 0.2)(1 - 0.45) = 0.8 \cdot 0.55 = 0.44$

(h) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.2 + 0.45 - 0.09 = 0.65 - 0.09 = 0.56$

part (a)

Group Exercise 3

What is the probability of getting three heads in three flips of a balanced coin?

let A be the event that 1st flip lands heads

let B ————— 2nd —————

let C ————— 3rd —————

$$\begin{aligned}\text{Then } P(A \text{ and } B \text{ and } C) &= P(A)P(B)P(C), \text{ since } A, B, \text{ \& } C \text{ are independent} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{2^3} = \frac{1}{8}\end{aligned}$$

Group Exercise 4

A jury consists of nine men and three women. If **three** of the jurors are randomly picked for an interview, what is the probability that they will all be women?

Let B be the event that the 1st two are women.

Let A be the event that the 3rd is a woman.

~~the~~ By the mult. rule, $P(A \text{ and } B) = P(B)P(A|B)$;

from ~~example~~ example 1 on pg. 5, $P(B) = \frac{1}{22}$.

If we had 12 jurors w. 3 women initially, and we remove 2 women from the group, then 10 remain with 1 woman,
so $P(A|B) = \frac{1}{10}$.

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Therefore, $P(A \cap B) = P(B)P(A|B) = \frac{1}{22} \cdot \frac{1}{10} = \frac{1}{220} \approx 0.005$
or 0.5%.

Remember the example about $P(\text{dark}|\text{midnight}) \neq P(\text{midnight}|\text{dark})$.

Group Exercise 5

Suppose that C is the event that a person committed a crime, and E is the event that a particular piece of evidence was found. State in words what $P(E|C)$ and $P(C|E)$ represent. How might this information be misunderstood by the general public?

$P(E|C)$ is the probability that evidence is found, given that a crime was committed

$P(C|E)$ — a crime was committed, given that evidence was found.

Trial lawyers & jurors confuse these all the time, leading to wrongful convictions.

Although these examples show us that $P(A|B) \neq P(B|A)$, we might be interested in the question of whether there is some other formula that relates $P(A|B)$ and $P(B|A)$.

Recall the two versions of the multiplication rule:

$$P(E \text{ and } F) = P(F) \cdot P(E|F)$$

and

$$P(E \text{ and } F) = P(E) \cdot P(F|E).$$

Notice that these two versions of the rule give two different ways of writing the same quantity $P(E \text{ and } F)$. So we can write:

$$P(F) \cdot P(E|F) = P(E) \cdot P(F|E),$$

which boils down to

Theorem 4 (Bayes' Theorem)

$$P(E|F) = \frac{P(E) \cdot P(F|E)}{P(F)}.$$

Bayes' Theorem is a result of Thm. 4 - we'll discuss next time.