Lecture 17:
Lesson and Activity Packet

MATH 232: Introduction to Statistics

December 5, 2016

Homework and Announcements

- Quiz 10 on Wednesday, December 7
- Last class on Monday, December 12 (one week from today)
- Homework 17 due in class Monday, December 12
- Extra Credit Opportunity due on Canvas at 11:59 p.m. on Monday, December 12 (you may also submit in hard copy during class that day)
- Post on book discussion forum on Canvas before 11:59 p.m. Monday
- Submit book summary 3 on Canvas before 11:59 p.m. Monday
- 10 a.m. section: your final exam is December 14, 10:30 a.m. in the usual classroom
- 12 p.m. section: your final exam is December 16, 1:00 p.m. in the usual classroom

Last time:

- Continuous random variables
- Normal distribution
- Area as probability

Questions?

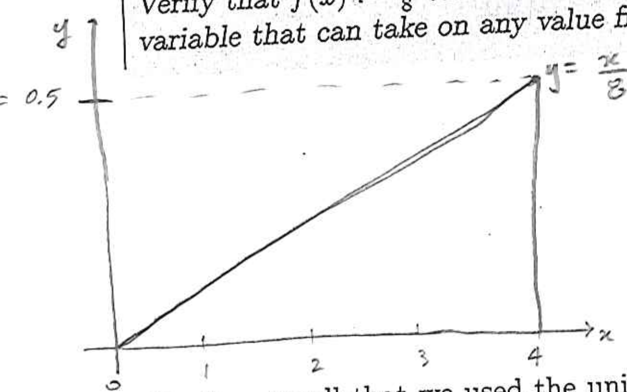
Today

- Normal distribution

Recall that the requirements of a density curve are that the area beneath the curve sum to 1, and that the curve never fall below the x-axis.

Group Exercise 1

Verify that $f(x) := \frac{x}{8}$ can serve as the probability density function for a continuous random variable that can take on any value from 0 to 4.



Check:

base, height

1. Area under curve $= \frac{1}{2}(4)\left(\frac{1}{2}\right) = \frac{4}{4} = 1$ ✓

2. Curve entirely above x-axis ✓

Further, recall that we used the uniform probability distribution to illustrate the notion of a correspondence between **area beneath the curve** and **probability**.

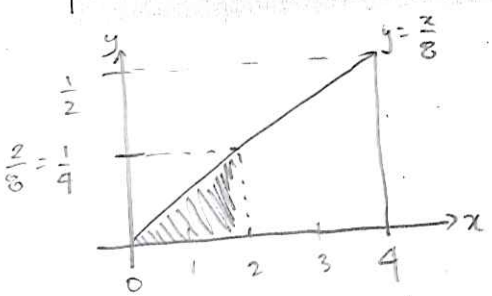
Definition 1 (*Probability*)

The area under the density curve between any two values a and b gives the probability that a random variable having the continuous distribution corresponding to that density curve will take on a value on the interval from a to b.

Group Exercise 2

With reference to the preceding exercise, find the probabilities that a random variable having the given probability density will take on a value:

- *less than 2;*
- *less than or equal to 2*



$P(x < 2) =$ Area of triangle shaded at left

$= \frac{1}{2} \cdot (2)\left(\frac{1}{4}\right) = \frac{2}{2 \cdot 4} = \frac{2}{8} = \frac{1}{4}$

$= 0.25 \doteq 25\%$

$P(x \le 2) = P(x < 2) + P(x = 2)$

$= 25\% + 0\%$

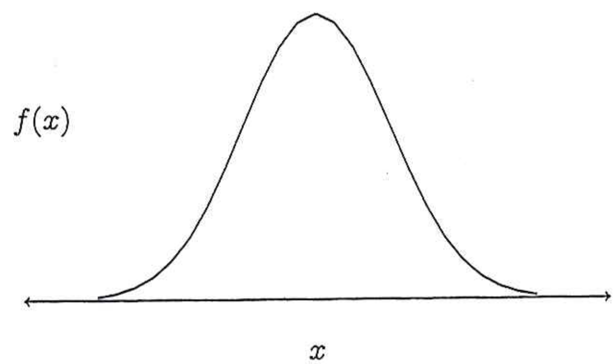
$= 25\%$

We learned last time about the uniformly distributed continuous random variables. This time, we'll learn about normally distributed continuous random variables.

Definition 2 (*Normal random variable*)

If a continuous random variable has a distribution with a graph that is symmetric and bell-shaped, as in the figure below, and if this bell shape can be described by the equation below, then we say that the random variable is normally distributed *or that it follows the* normal distribution.

$$y = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}$$



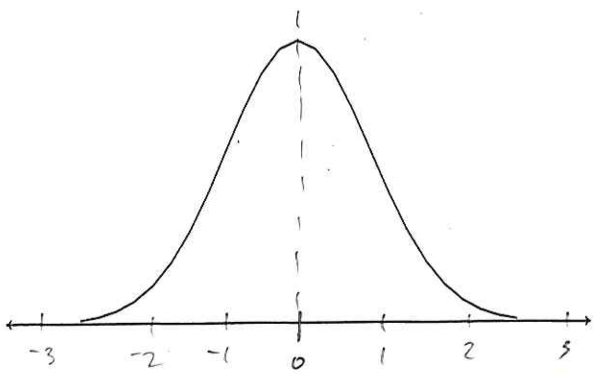
You do not need to memorize this formula, but look at it for a moment. It involves π and e, two irrational constants (their approximate values are 3.1415... and 2.71828..., respectively), and it involves σ and μ, the mean and standard deviation, respectively, of the random variable.

We see that the shape of the bell curve is entirely defined by the mean and standard deviation of the random variable it describes. Once we know μ and σ, we can draw the graph just like we would any other graph.

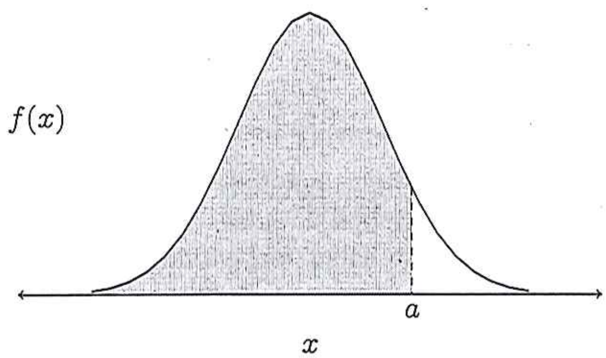
Note that the "tails" of the curve ~~to~~ extend to $\pm\infty$. There is **no end** to the range of values that this continuous random variable can take on, and that is okay: the area beneath the curve is still only 1. We will not prove this here, since it takes a great deal of mathematics to show.

Definition 3 (*Standard normal distribution*)

The standard normal distribution *is the normal probability distribution with* $\mu = 0$ *and* $\sigma = 1$*:*



Recall that for continuous random variables, the probability that $x = a$, for any a in the domain of x, is exactly 0. But we **do** have a way of defining the probability that $x < a$, that $x > a$, or that $a < x < b$. Just compute the area under the curve in that range. For example, the probability that $x < a$ or that $x \leq a$ is given by the shaded area:



Without the tools of calculus and analysis, there is no way of computing this area directly; however, there is a chart that you've been given on a separate piece of paper that gives a table of values that are useful for this. Some useful guidance and examples follow.

(last pg.)

Finding Probabilities When Given z Scores

Using Table A-2 (in Appendix A and the *Formulas and Tables* insert card), we can find areas (or probabilities) for many different regions. Such areas can also be found using a TI-83/84 Plus calculator, or computer software such as STATDISK, Minitab, or Excel. The key features of the different methods are summarized in Table 6-1 on the next page. Because calculators or computer software generally give more accurate results than Table A-2, we strongly recommend using technology. (When there are discrepancies, answers in Appendix D will generally include results based on Table A-2 as well as answers based on technology.)

If using Table A-2, it is essential to understand these points:

1. Table A-2 is designed only for the *standard* normal distribution, which has a mean of 0 and a standard deviation of 1.
2. Table A-2 is on two pages, with one page for *negative z* scores and the other page for *positive z* scores.

Note to Instructor:
Recommendation: Require or strongly encourage the drawing of a graph for each problem solved. Point out that the graph provides a visual understanding that can be really helpful when solving the problems of this important chapter. Also, constantly stress the difference between *areas* under the curve and z scores that are *distances* representing the number of standard deviations that a value is away from the mean.

3. Each value in the body of the table is a *cumulative area from the left* up to a vertical boundary above a specific z score.
4. When working with a graph, avoid confusion between z scores and areas.

   z score: ***Distance*** **along the horizontal scale of the standard normal distribution; refer to the leftmost column and top row of Table A-2.**

   Area: ***Region*** **under the curve; refer to the values in the body of Table A-2.**

5. The part of the z score denoting hundredths is found across the top row of Table A-2.

CAUTION

When working with a normal distribution, avoid confusion between z scores and areas.

TABLE A-2 (*continued*) Cumulative Area from the LEFT

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

The area (or probability) value of 0.8980 indicates that there is a probability of 0.8980 of randomly selecting a z score less than 1.27. (The following sections will consider cases in which the mean is not 0 or the standard deviation is not 1.)

EXAMPLE 3 **Scientific Thermometers** The Precision Scientific Instrument Company manufactures thermometers that are supposed to give readings of 0°C at the freezing point of water. Tests on a large sample of these instruments reveal that at the freezing point of water, some thermometers give readings below 0° (denoted by negative numbers) and some give readings above 0° (denoted by positive numbers). Assume that the mean reading is 0°C and the standard deviation of the readings is 1.00°C. Also assume that the readings are normally distributed. If one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is less than 1.27°.

SOLUTION The probability distribution of readings is a standard normal distribution, because the readings are normally distributed with $\mu = 0$ and $\sigma = 1$. We need to find the area in Figure 6-5 below $z = 1.27$. The *area* below $z = 1.27$ is equal to the *probability* of randomly selecting a thermometer with a reading less than 1.27°. From Table A-2 we find that this area is 0.8980.

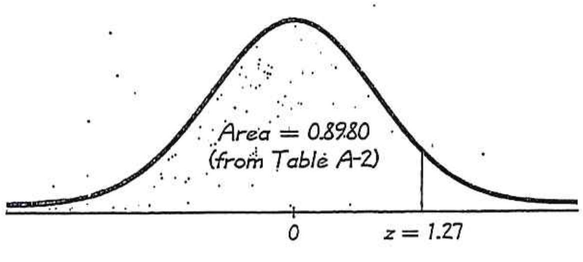
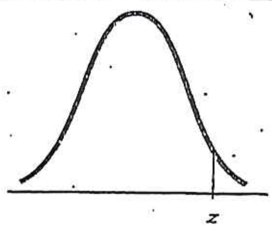
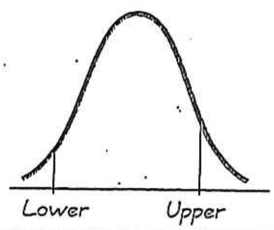


Figure 6-5
Finding the Area Below $z = 1.27$

INTERPRETATION The *probability* of randomly selecting a thermometer with a reading less than 1.27° (at the freezing point of water) is equal to the area of 0.8980 shown as the shaded region in Figure 6-5. Another way to interpret this result is to conclude that 89.80% of the thermometers will have readings below 1.27°.

Table 6-1 Methods for Finding Normal Distribution Areas

Table A-2, STATDISK, Minitab, Excel Gives the cumulative area from the left up to a vertical line above a specific value of z.		**Table A-2** The procedure for using Table A-2 is described in the text. **STATDISK** Select **Analysis, Probability Distributions, Normal Distribution.** Enter the z value, then click on **Evaluate.** **MINITAB** Select **Calc, Probability Distributions, Normal.** In the dialog box, select **Cumulative Probability, Input Constant.** **EXCEL** Select **fx, Statistical, NORMDIST.** In the dialog box, enter the value and mean, the standard deviation, and "true."
TI-83/84 Plus Calculator Gives area bounded on the left and bounded on the right by vertical lines above any specific values.		**TI-83/84** Press 2ND VARS **[2: normal cdf (]**, then enter the two z scores separated by a comma, as in (left z score, right z score).

The following example requires that we find the probability associated with a z score less than 1.27. Begin with the z score of 1.27 by locating 1.2 in the left column; next find the value in the adjoining row of probabilities that is directly below 0.07, as shown in the following excerpt from Table A-2.

NEGATIVE z Scores

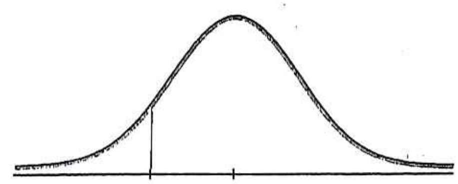


TABLE A-2 Standard Normal (z) Distribution: Cumulative Area from the LEFT

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.50 and lower	.0001									
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	* .0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	* .0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

NOTE: For values of z below −3.49, use 0.0001 for the area.
*Use these common values that result from interpolation:

z score	Area
−1.645	0.0500
−2.575	0.0050

TABLE A-3 t Distribution: Critical t Values

Degrees of Freedom	0.01	0.02	0.05	0.10	0.20
	Area in One Tail				
	0.005	0.01	0.025	0.05	0.10
	Area in Two Tails				
	0.01	0.02	0.05	0.10	0.20
1	63.657	31.821	12.706	6.314	3.078
2	9.925	6.965	4.303	2.920	1.886
3	5.841	4.541	3.182	2.353	1.638
4	4.604	3.747	2.776	2.132	1.533
5	4.032	3.365	2.571	2.015	1.476
6	3.707	3.143	2.447	1.943	1.440
7	3.499	2.998	2.365	1.895	1.415
8	3.355	2.896	2.306	1.860	1.397
9	3.250	2.821	2.262	1.833	1.383
10	3.169	2.764	2.228	1.812	1.372
11	3.106	2.718	2.201	1.796	1.363
12	3.055	2.681	2.179	1.782	1.356
13	3.012	2.650	2.160	1.771	1.350
14	2.977	2.624	2.145	1.761	1.345
15	2.947	2.602	2.131	1.753	1.341
16	2.921	2.583	2.120	1.746	1.337
17	2.898	2.567	2.110	1.740	1.333
18	2.878	2.552	2.101	1.734	1.330
19	2.861	2.539	2.093	1.729	1.328
20	2.845	2.528	2.086	1.725	1.325
21	2.831	2.518	2.080	1.721	1.323
22	2.819	2.508	2.074	1.717	1.321
23	2.807	2.500	2.069	1.714	1.319
24	2.797	2.492	2.064	1.711	1.318
25	2.787	2.485	2.060	1.708	1.316
26	2.779	2.479	2.056	1.706	1.315
27	2.771	2.473	2.052	1.703	1.314
28	2.763	2.467	2.048	1.701	1.313
29	2.756	2.462	2.045	1.699	1.311
30	2.750	2.457	2.042	1.697	1.310
31	2.744	2.453	2.040	1.696	1.309
32	2.738	2.449	2.037	1.694	1.309
33	2.733	2.445	2.035	1.692	1.308
34	2.728	2.441	2.032	1.691	1.307
35	2.724	2.438	2.030	1.690	1.306
36	2.719	2.434	2.028	1.688	1.306
37	2.715	2.431	2.026	1.687	1.305
38	2.712	2.429	2.024	1.686	1.304
39	2.708	2.426	2.023	1.685	1.304
40	2.704	2.423	2.021	1.684	1.303
45	2.690	2.412	2.014	1.679	1.301
50	2.678	2.403	2.009	1.676	1.299
60	2.660	2.390	2.000	1.671	1.296
70	2.648	2.381	1.994	1.667	1.294
80	2.639	2.374	1.990	1.664	1.292
90	2.632	2.368	1.987	1.662	1.291
100	2.626	2.364	1.984	1.660	1.290
200	2.601	2.345	1.972	1.653	1.286
300	2.592	2.339	1.968	1.650	1.284
400	2.588	2.336	1.966	1.649	1.284
500	2.586	2.334	1.965	1.648	1.283
1000	2.581	2.330	1.962	1.646	1.282
2000	2.578	2.328	1.961	1.646	1.282
Large	2.576	2.326	1.960	1.645	1.282