L10: Feb. 23, 2017.

Recap: Finding inverses.

A matrix $A \in \mathbb{R}^{m \times n}$ has a <u>left inverse</u>, denoted A_L^{-1}, if

$A_L^{-1} A = I_n$. [Note: $A_L^{-1} \in \mathbb{R}^{n \times m}$.]

A matrix $A \in \mathbb{R}^{m \times n}$ has a <u>right inverse</u>, denoted A_R^{-1},

if $AA_R^{-1} = I_m$. [Note: $A_R^{-1} \in \mathbb{R}^{n \times m}$.]

Q: Suppose tht. for $A \in \mathbb{R}^{m \times n}$, A_L^{-1} and A_R^{-1} both exist.

Is $A_L^{-1} = A_R^{-1}$ true?

$$A_L^{-1} A = I_n \quad , \quad \boxed{AA_R^{-1} = I_m .}$$

want to show: $A_L^{-1} = A_R^{-1}$

$(n \times n) \; (n \times m)$ ✓

$$\underbrace{A_L^{-1} A}A_R^{-1} = \underbrace{I_n A_R^{-1}}$$

$$A_L^{-1} I_m = A_R^{-1}$$

$I_n \cdot$ (anything with n rows) = that same "anything"

$A_L^{-1} I_m = A_L^{-1}$

$$A_L^{-1} = A_R^{-1}$$

Example. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. $A \in \mathbb{R}^{3\times 2}$.

Left inverse: Observe that $A_L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ gives

$$A_L^{-1} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Check: $$A A_L^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3.$$

- For our $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, we found a left inverse $A_L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

  We found $AA_L^{-1} \neq I_3$ — so the left inverse is <u>not</u> also the right inverse.

- If A_L^{-1} & A_R^{-1} both exist, then $A_L^{-1} = A_R^{-1}$.

  If $A_L^{-1} \neq A_R^{-1}$, then A_L^{-1} & A_R^{-1} do not both exist.

So: <u>A does not have a right inverse!</u>

In general, $A \in \mathbb{R}^{m \times m}$ is said to be invertible or non singular if $\exists A^{-1} \in \mathbb{R}^{m \times m}$ s.t. $AA^{-1} = I_m = A^{-1}A$.

[i.e., $A \in \mathbb{R}^{m \times m}$ is invertible if it has both a Ⓛ and a Ⓡ inverse.]

Finding matrix inverses:
- Augment w/ I_m
- Row-reduce.

If a matrix $A \in \mathbb{R}^{m \times m}$ is invertible, then it is row-equivalent to I_m.

Also, if $A \in \mathbb{R}^{m \times m}$ is invertible, then $A\vec{x} = \vec{b}$ has one unique sol'n for any $\vec{b} \in \mathbb{R}^m$.

In particular, the sol'n to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

$$\underbrace{A^{-1}A}\vec{x} = A^{-1}\vec{b}$$
$$I_m \vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}.$$

$$\left(A \,\middle|\, \vec{b}\right) \sim \left(\text{rref}(A) \,\middle|\, \text{modified } \vec{b}\right)$$

- if rref(A) has a row of zeros:

$$\left(\begin{array}{ccc|c} \sim\sim & & & * \\ \sim\sim & & & * \\ 0 & \cdots & 0 & \boxed{*} \end{array}\right)$$

$=0$: inf. many sol'ns

$\neq 0$: $\vec{0}$ sol'n.

- if $A \sim I$: $\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & 0 \cdots & 0 & * \\ 0 & \cdots & & 1 & * \end{array}\right)$ → unique sol'n

- If A is invertible, then the only soln to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

$$\underset{(n\times n)\ (n\times 1)}{A \cdot \vec{0}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{but also} \quad \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & & 1 & 0 \end{array}\right)$$

is the rref of the system.

When a matrix is invertible...

- $A\vec{x} = \vec{b}$ has a unique soln for ~~all~~ any $\vec{b}$ in $\mathbb{R}^n$
- $A\vec{x} = \vec{0}$ has only the trivial soln $\vec{x} = \vec{0}$.
- $\text{rref}(A) = I$.

If A is invertible (nonsingular), then is A^{-1} invertible?? (non-sg.)

i.e., $\overset{?}{\exists}\, (A^{-1})^{-1} \in \mathbb{R}^{n\times n}$ s.t. $(A^{-1})^{-1}A^{-1} = I_n = A^{-1}(A^{-1})^{-1}$?

Try $(A^{-1})^{-1} = A$: $\quad AA^{-1} = I_n$, because A^{-1} was the inverse of A.

$A^{-1}A = I_n$ ——— " ———

So, yes, $(A^{-1})^{-1} = A$ works!

THM: If A is non-sg., then A^{-1} is non-sg., and $(A^{-1})^{-1} = A$.

THM. If $A \in \mathbb{R}^{n\times n}$ is non-sg. and has inverse A^{-1}

and $B \in \mathbb{R}^{n\times n}$ —— " —————— B^{-1},

Then (AB) is non-sg., and has inverse $B^{-1}A^{-1}$.

Pf. $(AB)(B^{-1}A^{-1}) = I$: Because matrix mult. is associative, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$. Because B has inverse B^{-1}, $BB^{-1} = I$ (by the def'n of inverses), so $A(BB^{-1})A^{-1} = AIA^{-1} = (AI)A^{-1} = AA^{-1}$, because anything mult. by I is itself. Also, by the def'n of inverses, $AA^{-1} = I$.
So, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

$(B^{-1}A^{-1})(AB) = I$:

$$(B^{-1}A^{-1})(AB) \underset{\text{assoc.}}{=} B^{-1}(A^{-1}A)B \underset{\text{def'n of inv.}}{=} B^{-1}IB \underset{\text{def. of } I}{=} B^{-1}B \underset{\text{def'n of inv.}}{=} I.$$

THM. If $A \in \mathbb{R}^{n \times n}$ is invertible, then A^T is invertible, and $(A^T)^{-1} = $ ______ .

Pf.

§ $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Finding A^{-1}: $\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \underset{R2-2R1}{\overset{R1}{\sim}} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$.

Testing A^{-1}: $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$A^T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Finding $(A^T)^{-1}$: $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \underset{R2}{\overset{R1-2R2}{\sim}} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right)$.

Testing $(A^T)^{-1}$: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

So $A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ and $(A^T)^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

For this case, $(A^T)^{-1} = (A^{-1})^T$.

Is this true in general?

<u>Example.</u> $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

Finding A^{-1}: $\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \sim \begin{array}{l} R1 - \frac{3}{2}R_2 \\ \frac{1}{2}R_2 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -3/2 \\ 0 & 1 & 0 & 1/2 \end{array}\right)$

So $A^{-1} = \begin{pmatrix} 1 & -3/2 \\ 0 & 1/2 \end{pmatrix}$. ($\underline{\text{check}}$: $A^{-1}A \overset{?}{=} I \overset{?}{=} AA^{-1}$)

Finding $(A^T)^{-1}$: $\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \sim \begin{array}{l} R1 \\ R2-3R1 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{array}\right)$

$\sim \begin{array}{l} R1 \\ \frac{1}{2}R_2 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right)$

So $(A^T)^{-1} = \begin{pmatrix} 1 & 0 \\ -3/2 & 1/2 \end{pmatrix}$ ($\underline{\text{Check}}$: $(A^T)^{-1}A^T \overset{?}{=} I$

$=$ $A^T(A^T)^{-1} \overset{?}{=} I$)

$A^{-1} = \begin{pmatrix} 1 & -3/2 \\ 0 & 1/2 \end{pmatrix}$. $(A^{-1})^T = \begin{pmatrix} 1 & 0 \\ -3/2 & 1/2 \end{pmatrix}$