

L16: March 23, 2017.

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Housekeeping: WebWork due Saturday night, 11:59 p.m.

— " — Tuesday — " —

Written homework due Tuesday in class

Exam 2 on Thursday, March 28

Last time: Vector subspaces.

Questions?

This time: Row space of a matrix
Column space of a matrix

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graph LR; A[Row rank  
Column rank  
Nullity] --> B["(Null space of a matrix)"]
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Let $A \in \mathbb{R}^{m \times n}$, and let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}.$$

Then the column space of A is the vector subspace of \mathbb{R}^m that is spanned by the columns of A , i.e.,

$$\text{col}(A) = \text{span} \left(\left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \right).$$

The row space of A is the vector subspace of \mathbb{R}^n spanned by the rows of A , i.e.,

$$\text{row}(A) = \text{span} \left(\left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix}, \begin{bmatrix} a_{21} \\ \vdots \\ a_{2n} \end{bmatrix}, \dots, \begin{bmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \right)$$

in \mathbb{R}^n , $\begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} = [a_{11} \dots a_{1n}] = \langle a_{11}, \dots, a_{1n} \rangle$.

What makes the column space of an $m \times n$ matrix a vector subspace of \mathbb{R}^m ?

$\text{col}(A) \subseteq \mathbb{R}^m$ is a v. subsp. of \mathbb{R}^m if:

① $\text{col}(A)$ is closed under \wedge add'n vector

② $\text{col}(A)$ is closed under scalar mult.

① ~~Suppose~~ let $\vec{a}_1 := \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\vec{a}_2 := \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}$, ..., $\vec{a}_n := \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$.

Suppose $\vec{v}_1 \in \text{col}(A)$ and $\vec{v}_2 \in \text{col}(A)$.

Then $\exists c_1, c_2, \dots, c_n \in \mathbb{R}$, s.t.

$$\vec{v}_1 = c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n$$

and $\exists d_1, d_2, \dots, d_n \in \mathbb{R}$, s.t.

$$\vec{v}_2 = d_1 \vec{a}_1 + d_2 \vec{a}_2 + \dots + d_n \vec{a}_n.$$

observe: $\vec{v}_1 + \vec{v}_2 = (c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_m \vec{a}_m) + (d_1 \vec{a}_1 + d_2 \vec{a}_2 + \dots + d_m \vec{a}_m)$

$$= (c_1 + d_1) \vec{a}_1 + (c_2 + d_2) \vec{a}_2 + \dots + (c_m + d_m) \vec{a}_m.$$

Let $e_1 := c_1 + d_1$,
 $e_2 := c_2 + d_2$
 \vdots
 $e_m := c_m + d_m.$

Then $\vec{v}_1 + \vec{v}_2 = e_1 \vec{a}_1 + e_2 \vec{a}_2 + \dots + e_m \vec{a}_m.$

Since for $i \in [1, m] \cap \mathbb{N}$, $c_i \in \mathbb{R}$ and $d_i \in \mathbb{R}$, we have $e_i \in \mathbb{R}$ too.

Therefore, $\vec{v}_1 + \vec{v}_2 \in \text{span}(\{\vec{a}_1, \dots, \vec{a}_m\}) = \text{col}(A).$

② let $\vec{v} \in \text{col}(A)$, and let $c \in \mathbb{R}$. Then $\exists d_1, \dots, d_m \in \mathbb{R}$ s.t.

$$\vec{v} = d_1 \vec{a}_1 + d_2 \vec{a}_2 + \dots + d_m \vec{a}_m.$$

observe: $c\vec{v} = c(d_1 \vec{a}_1 + d_2 \vec{a}_2 + \dots + d_m \vec{a}_m)$

$$= cd_1 \vec{a}_1 + cd_2 \vec{a}_2 + \dots + cd_m \vec{a}_m.$$

Let $c_1 := cd_1$, $c_2 := cd_2$, ..., $c_m := cd_m$. Then

$$c\vec{v} = c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_m \vec{a}_m, \text{ and since}$$

$c \in \mathbb{R}$ and $\forall i \in [1, m] \cap \mathbb{N}$, $d_i \in \mathbb{R}$, we have $c_i \in \mathbb{R}$ too,

and so $c\vec{v} \in \text{col}(A).$

L16, cont'd.

Thm: If A and B are two row-equivalent matrices,
then $\text{row}(A) = \text{row}(B)$.

Pf. $\nexists A \ncong B$ are row-equivalent. Then B can be obtained from A by row-reducing—i.e., a finite number of elementary row operations (like row swaps, scaling, and combining rows).

Each elementary row operation consists of replacing a row of the matrix by a linear combination of rows of the matrix.

Therefore, each row of B is a lin. comb. of the rows of A . Therefore, $\text{row}_{\text{all}}(B) \subseteq \text{row}(A)$.

Since $A \ncong B$ were row-equivalent, A can also be obtained by row-reducing B (just undo all of the previous row ops).

So $\text{row}(A) \subseteq \text{row}(B)$.

Therefore, $\text{row}(A) = \text{row}(B)$. \square

Thm. If $\vec{b} \in \text{col}(A)$, then $A\vec{x} = \vec{b}$ has a sol'n, and
 If $A\vec{x} = \vec{b}$ has a sol'n, then $\vec{b} \in \text{col}(A)$.

i.e., $\vec{b} \in \text{col}(A)$ iff $A\vec{x} = \vec{b}$ has a sol'n.

Pf. (\Rightarrow) Suppose $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$, $\vec{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$, \dots , $\vec{a}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$

and suppose $\vec{b} \in \text{col}(A)$.

Therefore, $\exists x_1, \dots, x_n \in \mathbb{R}$ s.t.

$$\vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n.$$

Observe that if $\vec{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} | & | & \dots & | \\ a_1 & a_2 & \dots & a_n \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n \\ &= \vec{b}. \end{aligned}$$

So $\vec{x} := \langle x_1, \dots, x_n \rangle$ is a sol'n to $A\vec{x} = \vec{b}$.

(\Leftarrow) Suppose $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$, $\vec{b} \in \mathbb{R}^m$, and \vec{x} solves $A\vec{x} = \vec{b}$. Let $\vec{x} := \langle x_1, \dots, x_n \rangle$.

Observe: $A\vec{x} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n$,
 and so $\vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n$, so $\vec{b} \in \text{col}(A)$.

Linear Independence.

A set of vectors $V := \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is said t/b linearly independent if ~~the~~ the only set of constants c_1, c_2, \dots, c_m s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \vec{0}$ is $c_1=0, c_2=0, c_3=0, \dots, c_m=0$.

A set of vectors $V := \{\vec{v}_1, \dots, \vec{v}_m\}$ is said t/b linearly dependent if $\exists c_1, \dots, c_m \in \mathbb{R}$, not all zero, s.t. $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$.

Note: Linear (in)dependence are properties of sets of vectors.

To determine whether a set of vectors is linearly independent...

Ex. Is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ linearly independent?

(i.e., $\exists c_1, c_2 \in \mathbb{R}$, not both zero, s.t. $c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?)

Observe: $c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\exists \vec{c} \in \mathbb{R}^2$, s.t. $\begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$? Solve the linear system!

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right] \sim \begin{array}{l} R2 \\ R1+R2 \\ R3 \\ R4 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right] \sim \begin{array}{l} R1 \\ R3 \\ R3-R4 \\ R2+2R3 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \text{ so } c_1 = c_2 = 0.$$

So V is lin. INDEPENDENT.

Ex. Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ lin. dep. or lin. indep.?

$A =$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\begin{array}{l} R1 \\ R2 \\ R3-R1 \\ R4-2R1 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{l} R1 \\ R3 \\ R2-R3 \\ R4-2R3 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R1-R3 \\ R2 \\ R3 \\ R4-R3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

REF

There is a pivot in each column (can tell by REF or RREF), so \nexists free variables. So

the only soln to the 'homogeneous' eq'n $A\vec{x} = \vec{0}$ is the trivial soln $\vec{x} = \vec{0}$.

Therefore, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ is linearly independent.

Ex. Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

lin. dep., or lin. indep.?

$A := \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -2 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}$ has 3 rows, so can have

only 3 pivots (max.); since there are 4 cols, at least 1 col. has to be without a pivot; i.e., there is at least 1 free parameter in

solving $A\vec{x} = \vec{0}$. So $A\vec{x} = \vec{0}$ has a nontrivial

solution. So the cols. of A (i.e., our orig.

set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is lin. DEP.