Lecture 22: Oct. 26, 2016.

Housekeeping:

- Homework 4 due in class today
- Canvas HW due Friday 11:59 p.m.
- Written HW 5 due next Wednesday in class.

Last time: 1^{st} & 2^{nd} derivs. of param. curves

This time: Area btwn. param. curve & axis

Arc length

Warm up.

Find the points on the curve
where the tangent line
is horizontal.

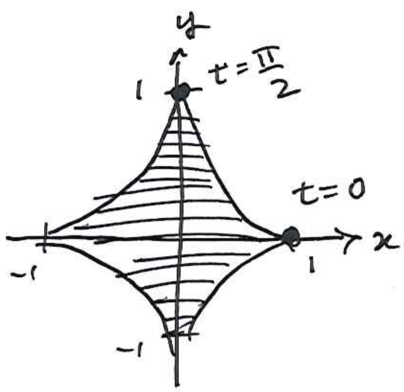
$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$$

Bonus: Find the points where the tan. line is vertical.

Area enclosed by graph of param. curve.

Example 1 Find the area enclosed by the

astroid
$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \\ 0 \le t \le 2\pi \end{cases}$$



Note: the astroid is symmetric, so the total area is 4 times the area beneath the curve in the 1^{st} quadrant. (The 1^{st} quadrant has $t \in [0, \pi/2]$.)

$$A = 4 \cdot \int_{x=0}^{1} y \, dx.$$

Note: $x = 0$ corresponds to $t = \frac{\pi}{2}$

$x = 1$ corr. to $t = 0$

$$A = 4 \cdot \int_{t=\frac{\pi}{2}}^{0} \sin^3(t) \cdot \left[\frac{dx}{dt} \, dt \right]$$

$$= 4 \cdot \int_{\pi/2}^{0} \sin^3(t) \left[\frac{d}{dt} \left[\cos^3(t) \right] \right] dt$$

$$= 4 \cdot \int_{\pi/2}^{0} \sin^3(t) \cdot 3\cos^2(t)(-\sin t) \, dt$$

$$= 4 \cdot 3 \int_{\pi/2}^{0} -\sin^4 t \cos^2 t \, dt$$

$$= 12 \int_{0}^{\pi/2} \sin^4 t \cos^2 t \, dt$$

$$= 12 \int_{0}^{\pi/2} \sin^4 t \left(1 - \sin^2 t\right) dt$$

$$= 12 \int_{0}^{\pi/2} \sin^4 t - \sin^6 t \, dt$$

double-angle formulas ?

$$A = 12 \cdot \frac{\pi}{32} = \boxed{\frac{3\pi}{8}}.$$

Length of a Parametrically Def. Curve.

Def. If C is defined by $\begin{cases} x = f(t) \\ y = g(t) \\ t \in [a,b] \end{cases}$,

where $f'(t)$ and $g'(t)$ are cts.① and not simultaneously zero on $[a,b]$②, and if C is traversed exactly③ once as t increases from a to b, then

the length of C is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE 2 $\begin{cases} x = r\cos t \\ y = r\sin t \\ 0 \le t \le 2\pi \end{cases}$ circle of radius r.

Check: ① $f'(t) = \frac{dx}{dt} = \frac{d}{dt}[r\cos t] = -r\sin t$ ✓

$g'(t) = \frac{dy}{dt} = \frac{d}{dt}[r\sin t] = r\cos t$ ✓

② $f'(t)$ & $g'(t)$ not simultaneously zero.

③ Is C traversed only once?

Ex 2 _ ct'd

So,

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\; dt$$

$$= \int_0^{2\pi} \sqrt{r^2 \underbrace{(\sin^2 t + \cos^2 t)}_{=1}}\; dt$$

$$= \int_0^{2\pi} \sqrt{r^2}\; dt \qquad (r > 0)$$

$$= \int_0^{2\pi} r\; dt$$

$$= r \cdot t \Big|_0^{2\pi} = r(2\pi - 0) = \boxed{2\pi r} \checkmark$$