Lecture 4:
Lesson and Activity Packet

MATH 330: Calculus III

September 16, 2016

Last time, we discussed:

- Series definition
  - Partial sums
  - Series convergence
  - Sum of series
  - Series divergence
- Telescoping series
- Geometric series
  - Repeating decimals

Questions on any of this?

If not, then today's lesson will be more about infinite series.

Last time, we learned that a series converges only when its sequence of partial sums converges. We also learned what a sequence of partial sums is:

Definition 1
For the series $\sum_{k=1}^{\infty} a_k$, the sequence of partial sums has terms S_n, where

$$\begin{aligned} S_1 &:= a_1 \\ S_2 &:= a_1 + a_2 \\ S_2 &:= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &:= a_1 + \cdots + a_n. \end{aligned}$$

Group Exercise 1 (*8 minutes*)
Prove that if the sequence $\{S_n\}$ of partial sums converges, then the sequence of terms $\{a_n\} \to 0$. Showing an example or two may help you figure out how this works, so try that approach if you're having trouble getting started[a].

[a] This is generally a good strategy for getting started proving things!

Note that for $n > 1$, $S_n = S_{n-1} + a_n$.

Then $a_n = S_n - S_{n-1}$. If $\{S_n\} \to S$, for example, then $\{a_n\} \to (S - S) = 0$.

What you just found in the last exercise should have been the following true statement:

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Group Exercise 2 (*5 minutes*)

Write the logical contrapositive of this statement. [See Packet 2 for a rundown on logical implication.]

If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 1 (*n^{th} Term Test for Divergent Series*)

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1

- $\sum_{n=1}^{\infty} \frac{n+1}{n}$ *diverges (as a series), because* $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.
- $\sum_{n=1}^{\infty} (-1)^n$ *diverges (as a series), because $\{(-1)^n\}$ does not converge at all, let alone converging to zero.*
- $\sum_{n=1}^{\infty} n^2$ *diverges (as a series), because $\{n^2\}$ diverges to infinity (it doesn't converge at all, let alone converging to 0).*

The n^{th} term test is a test for divergence, not convergence.

Example 2

One very important series to know about is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$.

The harmonic series does not converge. ***(The harmonic series does not converge.)*** *We do not yet have the tools to show this, but one of our goals over the next two classes will be to develop such tools.*

Group Exercise 3 (*3 minutes*)

Why is the harmonic series a counterexample to the converse of the n^{th} term test?

It has $\{\frac{1}{n}\} \to 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem 2 (*Combining Series*)
If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then:

- $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$,
- $\sum_{n=1}^{\infty} k \cdot a_n = kA$, *for any constant k,*
- $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$. *(follows from first two)*

Group Exercise 4 (*2 minutes*)
Prove that the third assertion follows from the first two.

If $\sum_{m=1}^{\infty} b_m = B$, then $\sum_{m=1}^{\infty} (-1) b_m = -B$ by Rule 2;

if $\sum_{m=1}^{\infty} (-1) b^m = B$ and $\sum_{m=1}^{\infty} a_m = A$, then $\sum_{m=1}^{\infty} a_m + (-1) b_m = A - B$ by Rule 1.

Group Exercise 5 (*7 minutes*)
Using the definition of series convergence from Packet 3 (the one that involves convergence of the sequence of partial sums), prove the first part of the theorem (that $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$).

If $\sum_{m=1}^{\infty} a_m = A$, then $\lim_{m \to \infty} \sum_{k=1}^{m} a_k = A$

and if $\sum_{m=1}^{\infty} b_m = B$, then $\lim_{m \to \infty} \sum_{k=1}^{m} b_k = B$.

Observe: $\sum_{k=1}^{m} a_m + b_m = \sum_{k=1}^{m} a_m + \sum_{k=1}^{m} b_m$ as a finite sum.

Then $\lim_{m \to \infty} \sum_{k=1}^{m} a_m + b_m = \lim_{m \to \infty} \left(\sum_{k=1}^{m} a_m + \sum_{k=1}^{m} b_m \right) = A + B$.

A corollary of Theorem 2 is:

Corollary 3 (*Corollary to Theorem 2*)
If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} k a_n$ also diverges.

Group Exercise 6 (*Optional Exercise, for those interested in logic*)
Show that ~~Corollary 1~~ follows from Theorem 2. You'll need to rewrite the if/then statement of Theorem 2 and take a contrapositive.

Another corollary of Theorem 1:

Corollary 4 (*Corollary to Theorem 2*)
If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n + b_n)$ both also diverge.

Group Exercise 7 (*5 minutes*)

We haven't had many examples of divergent series yet. Here's one: $\sum_{n=1}^{\infty} 1$. Write out the first four terms of this series.

$1 + 1 + 1 + 1 + \cdots$

Write out the first five partial sums of this series:

$\{1, 2, 3, 4, 5, \ldots\}$

What is the formula for the n^{th} partial sum of this series?

n

Does the sequence of partial sums converge or diverge?

Div. !

So now you can conclude that $\sum_{n=1}^{\infty} 1$ diverges.

Does the series $\sum_{n=1}^{\infty}(1 + \frac{1}{2^n})$ converge or diverge? Prove this.

$S_1 = 1 + \frac{1}{2}$

$S_2 = 1 + \frac{1}{2} + 1 + \frac{1}{4} = 2 + \frac{3}{4}$

$S_3 = 3 + \frac{7}{8}$

$S_4 = 4 + \frac{15}{16}$

$\vdots$

$S_n = n + \frac{2^n - 1}{2^n}$.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n + \frac{2^n-1}{2^n} \longrightarrow \infty$, so series diverges.

Group Exercise 8 (*10 minutes*)

*Does Corollary 2 say anything about the situation when $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ **both** diverge?*

That is, in the situation where $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, do we know anything about the convergence of $\sum_{n=1}^{\infty}(a_n + b_n)$?

Provide specific examples to back up your answer.

No. says nothing.

Example 1.

$a_n := 1$, $b_n := -1$.

Then $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} 1 + (-1) = \sum_{n=1}^{\infty} 0$,

which converges to 0.

but:

example 2.

$a_n := 1$, $b_n := 2$.

Then $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} 1 + 2 = \sum_{n=1}^{\infty} 3$

and partial sums are $S_n = 3n$,

which diverges, so series diverges.

So we have a theorem and two corollaries about combining series. To use those tools, we need to know some series and their convergence properties! We know about **telescoping series** and **geometric series**, some of which converge and some of which diverge. We also know about the harmonic series (**the harmonic series does not converge**). There are many more series on Page 419 of your textbook to use as examples. You don't need to memorize these, but some of them you can already work with. Their sums might be surprising.

Recap

- n^{th} term test for divergence
- Harmonic series
- Combining series
- Rehash of logical implication

Homework

- Canvas Homework 2 due 11:59 p.m. tonight.
- Canvas Homework 3 due 11:59 p.m. Monday.
- Written Homework 2 due at the beginning of class on Wednesday.