Lecture 7: Lesson and Activity Packet

MATH 330: Calculus III

September 28, 2016

Announcements and Homework

- Written Homework due in class one week from today
- Canvas Homework due Friday 11:59 p.m.

Recap

- Review of series until now
- Comparison test
- Limit comparison test
- Root test
- Ratio test

Questions on any of this?

If not, then today's lesson will be on **alternating series**.

In the preceding sections, we studied the root test and the ratio test, which are fundamentally different from the integral test, the comparison test, and the limit comparison test in that they apply even to series with negative terms.

There is a special kind of series whose terms are alternatingly positive and negative.

Definition 1 (*Alternating Series*)
An alternating series *is an infinite series of the form*

$$\sum_{n=0}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots,$$

where for all $n \in \mathbb{N}$, $a_n > 0$.

Example 1

- *The* alternating harmonic series *is* $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.
- *The geometric series* $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ *also alternates.*

Let's consider the partial sums of an alternating series in the particular case of an alternating series with $a_n \geq a_{n+1} > 0$ for all $n \in \mathbb{N}$ **and** $a_n \to 0$.

Theorem 1 (*Alternating Series Test*)

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *has* $a_n > 0$ *and satisfies the following two conditions:*

- $a_n \geq a_{n+1} > 0 \quad \forall n \in \mathbb{N}$, *and*
- $\lim_{n\to\infty} a_n = 0$

then the alternating series converges.

Example 2

Both the alternating harmonic series and the geometric series shown above converge, because the non-alternating parts of their terms decrease monotonically to zero.

This should be surprising! Although the harmonic series does not converge (the harmonic series does not converge), the *alternating* harmonic series **does** converge! At least there's something right and nice in the universe.

Group Exercise 1 (*2 minutes*)

Does the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ converge?

Well, $\lim_{n\to\infty}\frac{1}{2n-1} = 0$, and terms have non-alternating parts $\frac{1}{2n-1}$, which decrease to 0.

Group Exercise 2 (*5 minutes*)

What about $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2n-1}$? [Hint: Remember that the Alternating Series Test is **not** an "if-and-only-if"! You'll need to remember some other tools here.]

Note, $\lim_{n\to\infty}\frac{n}{2n-1} = \lim_{n\to\infty}\frac{1}{2-1/n} = \frac{1}{2} \neq 0$,

so the alt. Series test doesn't prove convergence.

However, try n^{th} term:

$\lim_{n\to\infty}\frac{(-1)^{n+1}n}{2n-1} = \lim_{n\to\infty}\frac{(-1)^n}{2-1/n}$ does not exist, as the odd terms approach $-\frac{1}{2}$ while even ones approach $\frac{1}{2}$. In particular, $\lim_{n\to\infty}\frac{(-1)^{n+1}n}{2n-1} \neq 0$, so series diverges by n^{th} term test.

Definition 2 (*Conditional convergence*)

If an alternating series $\sum a_n$ *converges, but* $\sum |a_n|$ *does not converge, then the series is called* conditionally convergent.

Definition 3 (*Absolute convergence*)

If the series $\sum |a_n|$ *converges, then the series* $\sum a_n$ *is called* absolutely convergent.

Example 3

The alternating harmonic series converges, but the (non-alternating) harmonic series does not. The alternating harmonic series is therefore conditionally convergent.

Group Exercise 3 (*3 minutes*)

Does the geometric series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$ converge absolutely, converge conditionally, or diverge?

Well, $\lim_{n\to\infty}\frac{1}{2^n} = 0$, so the alt. series converges.

But also $\sum_{n=0}^{\infty}\frac{1}{2^n}$ converges as a geometric series with $r = \frac{1}{2}$ so $|r| < 1$. Therefore, the alt. series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$ converges absolutely.

Recap of Ways to Determine Series Convergence

- Definition of partial sums and convergence an infinite series:

For the infinite series $\sum_{n=1}^{\infty} a_n$, the partial sums are defined by

$$S_1 = a_1; \quad S_2 = a_1 + a_2; \quad \cdots \quad S_N = \sum_{n=1}^{N} a_n \quad \cdots,$$

or equivalently by the recursion relation

$$S_1 = a_1; \quad S_n = S_{n-1} + a_n \text{ for } n \geq 2.$$

The series converges if and only if its sequence of partial sums converges, and if there is convergence, then the sum of the series is defined as the limit of the partial sums. It is particularly convenient to use this method for **telescoping series.**

- Geometric series:

$\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

- p-series:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

  - Special case: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Does not converge! (Does not converge!)

- Combining series:
  - If $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$.
  - If $\sum a_n = A$ then $\sum(ka_n) = kA$.
  - Every nonzero constant multiple of a divergent series diverges too.
  - If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge too.
  - If $\sum a_n$ and $\sum b_n$ *both* diverge, then we cannot say anything about either $\sum(a_n + b_n)$ or $\sum(a_n - b_n)$.
  - Adding or deleting finitely many terms doesn't affect whether a series converges, just (possibly) its sum.

  - Can reindex: $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n-1}$, and in general $\sum_{n=j}^{\infty} a_n = \sum_{n=j+h}^{\infty} a_{n-h}$.

- n^{th} term test for divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

  - Converse is false! Counterexample is the harmonic series, whose terms approach zero but which diverges anyway.

- Integral test

If $a_n = f(n)$ where f is positive, continuous, and decreasing for all $x, n \geq N \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\ dx$ either both converge, or both diverge (sometimes said: "the sum and the integral converge or diverge together").

- Comparison test

Suppose that for all $n > N$, a_n, b_n and c_n are all positive.

  - **If** $\sum c_n$ converges **and** for all $n > N$, $b_n \leq c_n$, **then** $\sum b_n$ converges too.
  - **If** $\sum a_n$ diverges **and** for all $n > N$, $a_n \leq b_n$, **then** $\sum b_n$ diverges too.

- Limit comparison test

Suppose $a_n > 0$ and $b_n > 0$ for all $n > N$.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ **and** $\sum b_n$ converges, then $\sum a_n$ converges too.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ **and** $\sum b_n$ diverges, then $\sum a_n$ diverges too.

- Ratio test

To determine convergence or divergence of the series $\sum a_n$, compute $L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

1. If $L < 1$, then the series converges;
2. If $L > 1$, then the series diverges;
3. If $L = 1$, then the test is **inconclusive.**

- Root test

To determine convergence or divergence of the series $\sum a_n$, compute $\rho := \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

1. If $\rho < 1$, then the series converges;
2. If $\rho > 1$, then the series diverges;
3. If $\rho = 1$, then the test is **inconclusive**.

- Alternating series test

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ has $a_n > 0$ and satisfies the following two conditions:

- $a_n \geq a_{n+1} > 0 \quad \forall n \in \mathbb{N}$, and
- $\lim_{n\to\infty} a_n = 0$

then the alternating series converges.