Lecture 8:
Lesson and Activity Packet

MATH 330: Calculus III

September 30, 2016

Announcements and Homework

- Written Homework due in class on Wednesday next week
- Canvas Homework due Monday 11:59 p.m.
- Exam 1 next Friday, October 7 (review day on Wednesday)

Recap

- Review of series until now
- Root test
- Ratio test
- Alternating series
- Absolute vs. conditional convergence

Questions on any of this?

If not, then today's lesson will be on **power series**.

Definition 1 (*Power series about $x = a$*)
A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots,$$

in which the center a is a constant, and the coefficients $c_0, c_1, \ldots$ may depend on n.

Notes:

- Often have $a = 0$: $\sum^{i} nfty_{n=0} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$
- Think of these like "infinite polynomials" —we will figure out how to manipulate them (e.g., adding, subtracting, differentiating, integrating) like "finite" polynomials

Example 1
If $a = 0$ and all coefficients c_n are 1, then the power series is

$$\sum_{n=0}^{\infty} x^n.$$

Group Exercise 1 (*5 minutes*)
The power series $\sum_{n=0}^{\infty} x^n$ is also classifiable as what kind of series? For which values of x does the series converge? What does it converge to in that case?

Geometric with $a=1$, $r=x$.

Converges if $|x|<1$, i.e., $x \in (-1,1)$.

In that case, conv. to $\frac{a}{1-r} = \frac{1}{1-x}$.

So we found:

$\boxed{\text{For } x \in (-1,1),\ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.}$

We can equivalently write:

$\boxed{\text{For } x \in (-1,1),\ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.}$

This latter representation might make it more clear that for $x \in (-1,1)$, a **polynomial approximation** of $\frac{1}{1-x}$ can be obtained by taking **partial sums** of the power series $\sum_{n=0}^{\infty} x^n$:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1+x \\ P_2(x) &= 1+x+x^2 \\ P_3(x) &= 1+x+x^2+x^3 \\ &\vdots \\ P_N(x) &= \sum_{n=0}^{N} x^n \\ &\vdots \end{aligned}$$

These polynomial approximations get closer to $\frac{1}{1-x}$ as $N \to \infty$.

Example 2

We know that $\frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$. *Watch the partial sums of the series* $\sum_{n=0}^{\infty} (\frac{1}{2})^n$ *get closer and closer to 2:*

$$\left\{P_0\left(\frac{1}{2}\right), P_1\left(\frac{1}{2}\right), P_2\left(\frac{1}{2}\right), P_3\left(\frac{1}{2}\right), \ldots, P_N\left(\frac{1}{2}\right), \ldots\right\} = \left\{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots, \frac{2^{N+1}-1}{2^N}, \ldots\right\}$$

Group Exercise 2 (*5 minutes*)

In the definition of the power series, take $a = 2$ and $c_n = (-\frac{1}{2})^n$:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = \sum_{n=0}^{\infty} \left(-\frac{x-2}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{2-x}{2}\right)^n.$$

For which values of x does this series converge? What does the series converge to, in that case?

Also a geom. series, $a=1$ and $r = \frac{2-x}{2}$, Conv. when $|r|<1$, i.e., $\left|\frac{2-x}{2}\right| < 1$, i.e., $|2-x| < 2$,

i.e., $2-x<2$ and $0-x > -2$
$x > 2-2 = 0$ and $x < 2+2 = 4$. So for $x \in (0,4)$.

If $x \in (0,4)$, $\sum_{n=0}^{\infty} \left(\frac{2-x}{2}\right)^n = \frac{a}{1-r} = \frac{1}{1-\frac{2-x}{2}} = \frac{1}{\frac{2-2+x}{2}} = \frac{2}{x}$.

Group Exercise 3 (*3 minutes*)

Give the general form of a polynomial approximation of the function $\frac{2}{x}$, and state where this approximation is valid.

$P_N(x) = \sum_{n=0}^{N} \left(\frac{2-x}{2}\right)^n$. Valid for $x \in (0,4)$.

We are often interested in the question: **For which values of x does a given power series converge?**

In each of the two previous examples/exercises, we saw that the power series centered at a converged for x **within a certain radius of** a. We'll see that this was no coincidence.

Theorem 1 (*Power Series Convergence*)

The convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is described by one of the following cases:

Case I. *There exists $R > 0$ such that the series converges absolutely for $|x-a| < R$ (i.e., converges for $x \in (a-R, a+R)$), but diverges if $|x-a| > R$ (i.e., diverges for $x \notin (a-R, a+R)$). At the endpoints $\{a-R, a+R\}$, the series* **may or may not** *converge absolutely; these must be tested individually.*

Case II. *The series converges absolutely for all x. In this case, we say $R = \infty$.*

Case III. *The series converges* **only** *at $x = a$, and diverges elsewhere. In this case, we say $R = 0$.*

Notes:

- R is called the **radius of convergence** and the interval where the series converges is called the **interval of convergence**. The interval of convergence may look like any of the following:
  1. $(a-R, a+R)$
  2. $(a-R, a+R]$
  3. $[a-R, a+R)$
  4. $[a-R, a+R]$
  5. $(-\infty, +\infty)$
  6. $\{a\}$

(3, ct'd from p.6) Then series diverges for all $x \neq 0$.
Interval is $\{0\}$ and radius is $R=0$.

(4) $L := \lim_{m\to\infty} \left| \frac{(-1)^{m+1} x^{2m+2} (2m)!}{(-1)^m x^{2m} (2m+2)!} \right| = \lim_{m\to\infty} \left| \frac{-x^2}{(2m+2)(2m+1)} \right| = 0$ for all x,

So by ratio test, int. of convergence of this series is $(-\infty, +\infty)$ and $R = \infty$.

Group Exercise 4 (*8 minutes*)

State the radius and interval of convergence for the following power series:

1. $\sum_{n=0}^{\infty} x^n$
2. $\sum_{n=0}^{\infty} \left(\frac{2-x}{2}\right)^n$
3. $\sum_{n=0}^{\infty} n^n x^n$ *[Hint: Use the ratio test.]*
4. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

(1) Saw in Ex. 1 tht. interval was $(-1, 1)$. $R=1$

(2) Saw in Ex. 2 tht. int. was $(0, 4)$. $R=2$

(3) Ratio: $\lim_{m\to\infty} \left| \frac{(m+1)^{m+1} x^{m+1}}{m^m x^m} \right| = \lim_{m\to\infty} \left| (m+1)\left(1+\frac{1}{m}\right)^m \cdot x \right| =$

However, $\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^m = \exp\left(\ln\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)\right) =$

$= \exp\left(\lim_{m\to\infty} m \cdot \ln\left(1+\frac{1}{m}\right)\right)$

$= \exp\left(\lim_{m\to\infty} \frac{\ln(1+1/m)}{1/m}\right)$

$= \exp\left(\lim_{m\to\infty} \frac{\left(\frac{1}{1+1/m}\right)(-1/m^2)}{(-1/m^2)}\right)$

$= \exp\left(\lim_{m\to\infty} \frac{1}{1+1/m}\right) = e.$

So $\lim_{m\to\infty} \left| (m+1)x\left(1+\frac{1}{m}\right)^m \right| = \lim_{m\to\infty} |x(m+1)| = \begin{cases} +\infty, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Convergence is important for multiplying power series, and for termwise differentiation and integration.

Theorem 2 (*Series multiplication for power series*)
If $A(x) := \sum a_n x^n$ and $B(x) := b_n x^n$ both converge absolutely for $|x| < R$,
Then for $c_n := \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$, the series $\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to $A(x)B(x)$ for $|x| < R$. That is,*

$$(\sum a_n x^n)(\sum b_n x^n) = \sum c_n x^n, \quad |x| < R.$$

Notes:

1. Take $R := \min\{R_a, R_b\}$, where R_a and R_b are the radii of convergence of $A(x)$ and $B(x)$ respectively.
2. Computing the c_n is often tedious!(!!) Can be easier to restrict the computation to the first few terms and use strategic $(\cdots)$'s.

Example. We know $\dfrac{1}{1-x} = \sum_{m=0}^{\infty} x^m$ for $x \in (-1,1)$

and if we are told that $e^x = \sum_{m=0}^{\infty} \dfrac{x^m}{m!}$ for all x,

then our theorem gives

$$\frac{e^x}{1-x} = \left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right)$$

$$= 1 + (1+1)x + \left(\tfrac{1}{2} + 1 + 1\right)x^2 + \left(\tfrac{1}{3!} + \tfrac{1}{2} + 1 + 1\right)x^3 + \cdots$$

$$= 1 + 2x + \tfrac{5}{2}x^2 + \tfrac{8}{3}x^3 + \cdots,$$

which is valid for $x \in (-1,1)$.

Theorem 3 (*Termwise differentiation of power series*)
If $\sum c_n (x-a)^n$ has a radius of convergence $R > 0$, then it defines a function $f(x) := \sum c_n (x-a)^n$ for x such that $|x - a| < R$, and:

$$f'(x) = \sum n c_n (x-a)^{n-1}$$
$$f''(x) = \sum n(n-1) c_n (x-a)^{n-2}$$
$$f'''(x) = \sum n(n-1)(n-2) c_n (x-a)^{n-3}$$
$$\vdots$$

Each of the series for the derivatives converges on the same interval, where $|x - a| < R$.

The general idea: if a power series converges, then it can be differentiated termwise on the interval of convergence.

Example 3
$f(x) := \frac{1}{1-x}$. We know that on $(-1,1)$,
$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m$. By our theorem, then, $f(x)$ is diff'ble on $(-1,1)$ and its derivative is

$$f'(x) = \sum_{m=0}^{\infty} m x^{m-1}.$$ Its 2^{nd} deriv. is

$$f''(x) = \sum_{m=0}^{\infty} m(m-1) x^{m-2}, \text{ etc.}$$

On the other hand, we know
$f'(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$ and $f''(x) = \frac{2}{(1-x)^3}$.
The series approximations conv. to their respective derivatives on $(-1,1)$.

This may **not** work for series that are not power series!

Example 4

$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ has terms strictly btwn. $\frac{-1}{n^2}$ and $\frac{1}{n^2}$, and both converge - so it converges.

But $\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2}$ fails the n^{th} term test, as

$\lim_{n\to\infty} \frac{n!\cos(n!x)}{n^2} = \infty \neq 0$, so it diverges.

Theorem 4 (*Termwise integration of power series*)

If $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then it defines a function $f(x) := \sum c_n(x-a)^n$ for x such that $|x-a| < R$, and $\int f(x)\,dx = \sum c_n \frac{(x-a)^{n+1}}{n+1} + C$ on the interval of convergence.

General idea: if a power series converges, then it can be integrated termwise on the interval of convergence.

Example. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ on $(-1,1)$, so

for $x \in (-1,1)$, $\int \frac{1}{1-x}\,dx = \sum_{n=0}^{\infty} \int x^n\,dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.

On the other hand, $\int \frac{1}{1-x}\,dx = -\ln|1-x|$. So

we know a p.s. expansion of $-\ln|1-x|$ on $(-1,1)$ is $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.