# Lecture 9: Lesson and Activity Packet

MATH 330: Calculus III

October 3, 2016

## Announcements and Homework

- Written Homework due in class on Wednesday
- Canvas Homework due tonight 11:59 p.m.
- Exam 1 now Wednesday, October 12 (review on Friday this week)

## Recap from last time

- Power series
- Radius and interval of convergence
- Multiplying power series
- Differentiating power series
- Integrating power series

Questions on any of this?

If not, then today's lesson will be on **Taylor series**, one particularly important kind of power series.



Recall that if a function $f(x)$ can be expressed as a power series centered at $x = a$ that has radius $R$ of convergence, then $f(x)$ is infinitely differentiable, and the finite sum

$$P_N(x) := \sum_{n=0}^{N} c_n(x-a)^n$$

is a polynomial that approximates $f(x)$ more and more closely as $N \to \infty$.

Now, for something much stronger:

**Theorem 1**

***Every function** $f(x)$ that is **infinitely differentiable** at $x = a$ can be written as a power series (with a nonzero radius of convergence).*

**Definition 1** (*Taylor series generated by $f$ at $x = a$*)

*Let $f(x)$ be infinitely differentiable in some neighborhood of $x = a$ (that is, in some open interval $(a-R, a+R)$). Then the* Taylor series generated by $f$ at $x = a$ *is:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots.$$

*If $a = 0$, we refer to the series as the* Maclaurin series generated by $f$.

**Example 1**

*Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where (if anywhere) does it converge to $1/x$?*

$f(x) = \frac{1}{x}$, so $f(2) = \frac{1}{2}$

$f'(x) = -\frac{1}{x^2}$, so $f'(2) = \frac{-1}{2^2} = \frac{-1}{4}$.

$f''(x) = \frac{2}{x^3}$, so $f''(x) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$

$f'''(x) = \frac{(-3)\cdot 2}{x^4}$, so $f'''(2) = \frac{-3\cdot 2}{2^4} = \frac{-3\cdot 2}{16} = \frac{-3}{8}$

$f^{(4)}(x) = \frac{4\cdot 3\cdot 2}{x^5}$, so $f^{(4)}(2) = \frac{4\cdot 3\cdot 2}{32} = \frac{3}{4}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$, so $f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}$.

Thus, Taylor series is $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^{n+1}} \left(\frac{1}{n!}\right) (x-2)^n =$

$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2-x}{2}\right)^{n+1}$. This is a

geom. series with $a = \frac{1}{2}$ and $r = \frac{2-x}{2}$, so

$|r| = \left|\frac{2-x}{2}\right|$ and $|r| < 1$ when $x \in (0,4)$ (see previous packet).

In that case, sum $= \frac{a}{1-r} = \frac{1/2}{1-\frac{2-x}{2}} = \frac{1/2}{\frac{2-2+x}{2}} = \frac{1}{x}$.

**Definition 2** (*Taylor polynomials*)

A Taylor polynomial of order $N$ is (usually) the $N^{th}$ partial sum of the Taylor series:

$$P_N(x) := \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N.$$

**Example 2**

*Find the Taylor series ~~and the fourth-order Taylor polynomial~~ generated by $f(x) = e^x$ at $x = 0$. [This is the same question as "Find the Maclaurin series generated by $f(x) = e^x$~~, and the fourth-order Taylor polynomial approximation of f about x = 0.~~"]*

$f(x) = e^x$, so $f(0) = e^0 = 1$

$f'(x) = e^x$, so $f'(0) = e^0 = 1$

$\vdots$

$f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{(1)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This conv. to $e^x$ where $e^x$ is infinitely diff'ble around $x=0$ — i.e., for all $x \in \mathbb{R}$.

**Group Exercise 1** (*10 minutes*)

*Show that the Maclaurin series generated by $f(x) :=$ $\cos(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Where does this series converge to $f$?*

$f(x) = \cos(x)$, so $f(0) = 1$

$f'(x) = -\sin(x)$, so $f'(0) = -\sin(0) = 0$

$f''(x) = -\cos(x)$, so $f''(0) = -\cos(0) = -1$

$f'''(x) = \sin(x)$, so $f'''(0) = \sin(0) = 0$

$f^{(4)}(x) = \cos(x)$, so $f^{(4)}(0) = \cos(0) = 1$.

Thus, $f^{(m)}(x) = \begin{cases} \cos x, & m \equiv 0 \pmod 4 \\ -\sin x, & m \equiv 1 \pmod 4 \\ -\cos x, & m \equiv 2 \pmod 4 \\ \sin x, & m \equiv 3 \pmod 4 \end{cases}$

and $f^{(m)}(0) = \begin{cases} 1, & m \equiv 0 \\ 0, & m \equiv 1 \\ -1, & m \equiv 2 \\ 0, & m \equiv 3 \end{cases} \pmod 4$.

So Maclaurin series is:

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

$$= 1 + 0 \cdot x - \frac{x^2}{2} + 0 \cdot x^3 + \frac{x^4}{4!} + 0 \cdot x^5 - \frac{x^6}{6!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Also, $$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = \frac{(-1)^0 \cdot x^{2 \cdot 0}}{(2 \cdot 0)!} + \frac{(-1)^1 x^{2 \cdot 1}}{(2 \cdot 1)!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

On the previous page, we showed that $f(x) = \cos(x)$ has the following derivatives, with their evaluations at $x = 0$:

$$\begin{aligned} f(x) &= \cos(x) & f(0) &= 1 \\ f'(x) &= -\sin(x) & f'(0) &= 0 \\ f''(x) &= -\cos(x) & f''(0) &= -1 \\ f'''(x) &= \sin(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \\ &\vdots \end{aligned}$$

The polynomial approximations are therefore:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 + 0 = 1 \\ P_2(x) &= 1 - \frac{x^2}{2} \\ P_3(x) &= 1 - \frac{x^2}{2} + 0 = 1 - \frac{x^2}{2} \\ P_4(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} \\ P_5(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + 0 = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \\ &\vdots \end{aligned}$$

Notice that the $N^{\text{th}}$ order (not "degree"!) Taylor polynomial approximation of $\cos(x)$ is **not** the same as the $N^{\text{th}}$ partial sum of the Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, because in this representation, the series "skips" the odd-numbered terms whose value is zero. Be careful!

**Group Exercise 2**

*Find the Taylor polynomials of order 1, 2, and 3 for $f(x) := \sin(x)$ about $x = 0$.*

$f(x) = \sin(x)$, so $f(0) = \sin(0) = 0$

$f'(x) = \cos(x)$, so $f'(0) = \cos(0) = 1$

$f''(x) = -\sin(x)$, so $f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos(x)$, so $f'''(0) = -\cos(0) = -1$

First order is $P_1(x) = x$

$2^{nd}$ order is $P_2(x) = x + 0\frac{x^2}{2!} = x$

$3^{rd}$ order is $P_3(x) = x + 0\frac{x^2}{2!} - \frac{x^3}{3!} = x - \frac{x^3}{6}$.



**Group Exercise 3**

*Find the Taylor series generated by $f(x) := \frac{1}{x^2}$ about $x = 1$.*

$f(x) = \frac{1}{x^2}$

$f'(x) = \frac{-2}{x^3}$

$f'''(x) = \frac{3 \cdot 2}{x^4}$

$f^{(4)}(x) = \frac{-4 \cdot 3 \cdot 2}{x^5}$

$\vdots$

$f^{(m)}(x) = \frac{(-1)^{m+1} m!}{x^{m+1}}$, so $f^{(m)}(1) = (-1)^{m+1} m!$

So T. series is

$$\frac{1}{x^2} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} m!}{m!}(x-1)^m = \sum_{m=0}^{\infty} (-1)^{m+1}(x-1)^m.$$