L6: February 7, 2017.

Housekeeping.

- Due today: Homework 5 (varying Δt and seeing the effects)
- Due Thursday: Homework 6 (graphing / plotting your System Dynamics results, and comparing to analytical solution)

Last time: Difference equation & a finite difference method

Questions?

Today: Unconstrained decay

Newton's Law of Heating + Cooling

Unconstrained decay.

The rate of change of the mass of a radioactive substance is proportional to the mass of the substance, with a negative proportionality constant.

For example: For carbon-14, the proportionality constant is approximately -0.00120968, which yields the differential equation:

$$\frac{dQ}{dt} = -0.00120968\, Q .$$

$\frac{dP}{dt} = kP$

$P(t) = P_0 e^{kt}$

As we saw last time, the solution is

$$Q(t) = \boxed{Q_0 \exp(-0.00120968\, t)}, \quad Q_0 \in \mathbb{R} .$$

plug this Q into LHS of ODE

Check: $$\frac{dQ}{dt} = \frac{d}{dt}\left[Q_0 \exp(-0.00120968\, t)\right]$$
$$= -0.00120968 \boxed{Q_0 \exp(-0.00120968\, t)}$$
$$= \underbrace{-0.00120968\, Q}_{\text{get RHS of ODE back — we're happy :)}} .$$

So, for carbon-14, $Q(t) = Q_0 \exp(-0.000120968\, t)$.

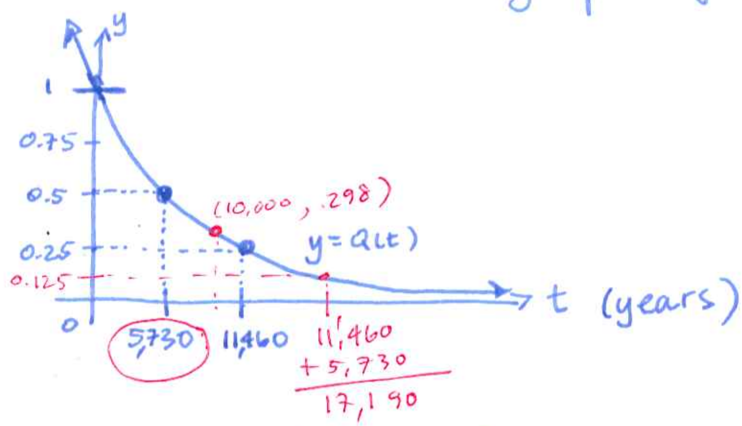
- What proportion of the original amount remains after 10,000 years?
  is $Q(0)$

$$\frac{Q(10{,}000)}{Q(0)} = \frac{Q_0 \exp(-0.000120968 \cdot 10{,}000)}{\underbrace{Q_0 \exp(\underbrace{-0.000120968 \cdot 0}_{0})}_{1}} = \frac{\cancel{Q_0} \exp(-1.20968)}{\cancel{Q_0}}$$

$$= \exp(-1.20968)$$

$$\approx 0.2982 = 29.8\%.$$

- Assume $Q_0 = 1$. Then the graph of $Q(t)$ looks like:



- What is the half-life of carbon-14?
  → amount of time it takes to decay to half its original quantity.

- What's the 75%-life?

What is the t-value s.t. $\frac{Q(t)}{Q(0)} = 0.75$?

$$\exp(-0.000120968\, t) = \frac{3}{4}$$

$$t = \frac{-\ln(3/4)}{0.000120968} = \frac{\ln(4/3)}{0.000120968}$$

$$\approx 2{,}378 \text{ (years)}.$$

Playing ....

If t_1 is the half-life of carbon-14, and we let $k := 0.000120968$, then

$$Q(t_1) = Q_0 \exp(-k t_1) = \tfrac{1}{2} Q_0 \,, \quad \text{so} \quad \underbrace{\exp(-k t_1) = \tfrac{1}{2}}.$$

question:

Does $Q(\tfrac{1}{2} t_1) \overset{?}{=} 0.75\, Q_0$

$$Q(\tfrac{1}{2} t_1) = Q_0 \exp(-\tfrac{1}{2} k t_1)$$

$$= Q_0 \big(\underbrace{\exp(-k t_1)}\big)^{1/2}$$

$$e^{ab} = (e^a)^b = (e^b)^a$$

$$= Q_0 \left(\tfrac{1}{2}\right)^{1/2} = Q_0 \sqrt{\tfrac{1}{2}}\,.$$

Since $\sqrt{\tfrac{1}{2}} \neq \tfrac{3}{4}$, $\tfrac{1}{2} t_1$ is not the $\tfrac{3}{4}$-life.

Example. You measure the amount of carbon-14 in a mummified body, and you find that there is only about 20% of the usual amount in a live body. HOW OLD ~~WAS~~ IS THE MUMMY YOU'RE STUDYING?

Find t_3 s.t. $Q(t_3) = 0.20\,Q_0$; i.e., find t_3 s.t.

$e^{-kt_3} = 0.20$. So $t_3 = \frac{\ln(0.20)}{-k} = \frac{\ln(5)}{k}$

$\approx 13,304.7$ years.

Problem. The half-life of radioactive strontium-90 is 29 years. Give the function that determines the amount present at time t.

Let k be the constant rate of decay; the general model is $Q(t) = Q_0 e^{-kt}$. Specifically, we know that

$$\frac{Q(29)}{Q(0)} = \frac{Q_0 e^{-29k}}{Q_0 e^{-0k}} = e^{-29k} = \frac{1}{2}.$$

Therefore, $e^{-29k} = \frac{1}{2}$, so $-29k = \ln(\frac{1}{2})$, or

$29k = \ln(2)$, and $k = \frac{\ln(2)}{29} \approx 0.0239$.

So $Q(t) = Q_0 e^{-\left(\frac{\ln(2)}{29}\right)t} = Q_0\left(e^{\ln(2)}\right)^{-\frac{t}{29}} = \boxed{Q_0 \cdot 2^{-t/29} = Q(t).}$

$$Q_0 \cdot 2^{-t/29} = Q(t)$$

$$\boxed{Q_0 \left(\tfrac{1}{2}\right)^{t/29} = Q(t)}$$

The proportion of strontium – 90 remaining after time t years is $\left(\frac{1}{2}\right)^{t/29}$.

- One more question... for Carbon-14, $Q(t) = Q_0 \exp(-0.000120968\, t)$.

What is the continuous decay rate in % per year?

$$\text{\% remaining after 1 year} = \frac{Q(1)}{Q(0)} = \frac{\cancel{Q_0}\, e^{-0.000120968}}{\cancel{Q_0}} = e^{-0.000120968}$$

$$\text{\% decayed after 1 year} = 1 - \text{\% remaining after 1 yr.} = 1 - e^{-0.000120968}$$

$$\approx 0.000120960\underline{68}$$

$$= 0.012096068 \text{ \% p.a.}$$

Newton's law of heating + cooling.

The rate of change of the temperature T of an object with respect to time t is proportional to the difference between the temp. of the surroundings (ambient temp.) and the temp. of the object.

The hotter an object, the <u>faster</u> it cools.
The colder an item, the <u>faster</u> it warms.

Suppose the ambient temperature is T_*. constant. Writing the equation...

$$\frac{dT}{dt} = k\left(T_* - T\right) \quad , \quad k > 0.$$

$$\frac{dT}{dt} = kT_* - kT(t)$$

$$\frac{dT}{dt} + kT(t) = kT_*.$$

$$e^{kt}\left(\frac{dT}{dt} + kT(t)\right) = kT_* e^{kt}$$

$$\frac{dT}{dt}e^{kt} + ke^{kt}T(t) = kT_* e^{kt}$$

$$\underbrace{\frac{dT}{dt}e^{kt} + \frac{d}{dt}\left[e^{kt}\right]T(t)} = kT_* e^{kt}$$

$$\frac{d}{dt}\left[T(t)e^{kt}\right] = kT_* e^{kt}$$

Aside: Use the prod. rule to compute $\frac{d}{dt}\left[T(t)e^{kt}\right]$.

$$\frac{d}{dt}\left[T(t)e^{kt}\right] = \frac{d}{dt}\left[T(t)\right]e^{kt} + T(t)\frac{d}{dt}\left[e^{kt}\right]$$

$$= \frac{dT}{dt}e^{kt} + \frac{d}{dt}\left[e^{kt}\right]T(t)$$

$$\frac{d}{dt}\left[T(t)e^{kt}\right] = kT_* e^{kt}$$

$$\int \frac{d}{dt}\left[T(t)e^{kt}\right]dt = \int kT_* e^{kt}\,dt$$

Antidifferentiati

FTC

$$T(t)e^{kt} = kT_*\left(\frac{1}{k}e^{kt}\right) + C$$

check RHS: $\frac{d}{dt}\left[\frac{kT_*}{k}e^{kt}\right]$

$= T_* k e^{kt}$

$$T(t)e^{kt} = \cancel{k}T_* e^{kt} + C$$

$$T(t) = \left(T_* e^{kt} + C\right)e^{-kt}$$

$$T(t) = T_* e^{kt}e^{-kt} + Ce^{-kt}$$

$$\boxed{T(t) = T_* + Ce^{-kt}}.$$

Check:

$$T(t) = T_* + Ce^{-kt}$$

$$\frac{dT}{dt} = k(T_* - T)$$

Plug into LHS:

$$\frac{dT}{dt} = \frac{d}{dt}\left[T_* + Ce^{-kt}\right]$$

$$= -kCe^{-kt}$$

Plug into the RHS of ODE:

$$k(T_* - T) = k\left(T_* - (T_* + Ce^{-kt})\right)$$

$$= k\left(T_* - T_* - Ce^{-kt}\right)$$

$$= k(-Ce^{-kt})$$

$$= -kCe^{-kt}$$

So, ODE is satisfied! :)