Final lecture: Thursday, April 27.

Nondimensionalization and the Buckingham Pi theorem.

We know basic notions of dimensional analysis already — for example, using conversion factors to convert quantities from one unit to another. And we always CHECK OUR UNITS!

Sometimes in mathematical modelling, dimensional analysis can provide surprisingly useful results ...

How to NONDIMENSIONALIZE an equation.

Illustration/example: the PROJECTILE PROBLEM: a body of constant mass m is radially projected upward from the earth's surface with initial speed V. Let R denote the radius of the Earth, and let $x(t)$ denote radial distance from the earth's surface at time t.

Neglecting air resistance, the governing differential eq'n and initial cond'ns are:

$$a = \frac{F_g}{m} = \frac{\text{"}g\text{"}\,\cancel{M}}{\text{"}R\text{"}^2}$$

$$\frac{R}{(x+R)}$$

$$\begin{cases} \dfrac{d^2x}{dt^2} = \dfrac{-gR^2}{(x+R)^2} \\[2ex] x(0) = 0 \\[2ex] \left.\dfrac{dx}{dt}\right|_{t=0} = V \end{cases},$$

where g is the gravitat'l acceleration on Earth.

To put the eq'n / problem

$$\begin{cases} d^2x/dt^2 = -gR^2/(x+R)^2 \\ x(0)=0, \quad \left.\dfrac{dx}{dt}\right|_{t=0} = V \end{cases} \qquad (*)$$

into nondimensional form, we follow a three-step process . . .

STEP ① : List all parameters & variables, together with their dimensions.

NOTE: Dimensions are different from units — the fundamental units are mass $\mathcal{M}$, length $\mathcal{L}$, time $\mathcal{T}$, and sometimes others such as charge $\mathcal{Q}$, etc., temp. Θ

Variables	Dimension
t	$\mathcal{T}$
x	$\mathcal{L}$

Parameters	Dimension
g	$\mathcal{L}\mathcal{T}^{-2}$
R	$\mathcal{L}$
V	$\mathcal{L}\mathcal{T}^{-1}$

not "the"

STEP ②: Let v^* be a variable; form a combination p^* of parameters with the same dimensions as v^*. Introduce $\frac{v^*}{p^*}$ as a new dimensionless variable. Do this for all variables v^*.

Variable x has dimension $\mathcal{L}$, the same dimension as parameter R. So let the new DIMENSIONLESS variable be

$$\boxed{\begin{aligned} x^* &:= \frac{x}{R}. \\ \Rightarrow \quad x &= R x^* \end{aligned}}$$

"scale x by R"
" — " — on R"

Variable t has dimension $\mathcal{T}$; note that the combination of parameters $\frac{R}{V}$ has dimension

$$\left[\frac{R}{V}\right] = \frac{[R]}{[V]} = \frac{\cancel{\mathcal{L}}}{\cancel{\mathcal{L}}\mathcal{T}^{-1}} = \mathcal{T},$$

just the same as t. So,

$$\text{let} \quad t^* := \frac{t}{R/V} = t\left(\frac{V}{R}\right).$$

"$\frac{R}{V}$ is the scaling factor for time"

STEP (3): Substitute the new variables into the problem.

The temptation is to say $\cancel{\frac{d^2x}{dt^2} = \frac{d^2x^*}{dt^{*2}}}$, but this is NOT VALID. Must use the CHAIN RULE:

$x = Rx^*$

$$\frac{dx}{dt} = \frac{dx}{dt^*}\frac{dt^*}{dt} = \frac{d}{dt^*}\left[x^* R\right]\cdot\frac{d}{dt}\left[t\left(\frac{V}{R}\right)\right] = \frac{dx^*}{dt^*}\cancel{R}\frac{V}{\cancel{R}} = \frac{dx^*}{dt^*}V.$$

$t^* = t\left(\frac{V}{R}\right)$

chain rule

Then... $$\frac{d^2x}{dt^2} = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{d}{dt^*}\left[\frac{dx}{dt}\right]\frac{dt^*}{dt} =$$

$$= \frac{d}{dt^*}\left[V\frac{dx^*}{dt^*}\right]\frac{d}{dt}\left[t\left(\frac{V}{R}\right)\right]$$

$$= \frac{d^2x^*}{dt^{*2}}\cdot V\cdot\frac{V}{R} = \boxed{\frac{V^2}{R}\frac{d^2x^*}{dt^{*2}} = \frac{d^2x}{dt^2}}$$

We substitute this into the equation & initial conditions, along with: $x = Rx^*$, $t^* = t\left(\frac{V}{R}\right)$.

$$\begin{cases} \dfrac{d^2x}{dt^2} = \dfrac{-gR^2}{(x+R)^2} \\ x(0) = 0 \\ \left.\dfrac{dx}{dt}\right|_{t=0} = V \end{cases}$$

becomes...

$t=0$ implies $t^* = 0\left(\frac{V}{R}\right) = 0$

$x=0$ implies $x^* = \frac{0}{R} = 0$

if $\frac{dx}{dt} = V$, then $\frac{dx^*}{dt^*} = \frac{1}{V}\frac{dx}{dt} = \frac{1}{V}(V) = 1$

$t=0$ implies $t^*=0$

$$\begin{cases} \dfrac{V^2}{R}\dfrac{d^2x^*}{dt^{*2}} = \dfrac{-gR^2}{(Rx^*+R)^2} \\ x^*(0) = 0 \\ \left.\dfrac{dx^*}{dt^*}\right|_{t^*=0} = 1 \end{cases}$$

Manipulation of the equation yields:

$$\frac{V^2}{R}\frac{d^2x^*}{dt^{*2}} = \frac{-gR^2}{(Rx^*+R)^2} = \frac{-gR^2}{[R(x^*+1)]^2} = \frac{-gR^2}{R^2(x^*+1)^2} = \frac{-g}{(x^*+1)^2}$$

so, $\frac{V^2}{R}\frac{d^2x^*}{dt^{*2}} = \frac{-g}{(x^*+1)^2} \Rightarrow \boxed{-\frac{V^2}{Rg}}\frac{d^2x^*}{dt^{*2}} = \frac{1}{(x^*+1)^2}$

$\left[\frac{-V^2}{Rg}\right] = \frac{[V]^2}{[R][g]} = \frac{\cancel{\mathcal{L}^2}\,\cancel{\mathcal{T}^{-2}}}{\cancel{\mathcal{L}}\,\cancel{\mathcal{L}}\,\cancel{\mathcal{T}^{-2}}} = 1$ $\rightarrow$ dimensionless constant

let $\varepsilon := \frac{-V^2}{Rg}$, so

$$\boxed{\varepsilon\frac{d^2x^*}{dt^{*2}} = \frac{1}{(x^*+1)^2}}$$

Remark: when we set $x^* := \frac{x}{R}$, R is called an intrinsic reference length, and when we set $t^* := t\frac{V}{R}$ $= t/_{R/V}$, R/V (what t is divided by) is called an intrinsic reference time.

In general, intrinsic reference quantities are defined to be standards of measurement formed from the parameters of a given problem. There might be different intrinsic reference quantities for a given problem — that's okay!

For example, $\sqrt{R/g}$ has dimensions $\sqrt{\mathcal{L}/\mathcal{L}\mathcal{T}^2} = \sqrt{\mathcal{T}^2} = \mathcal{T}$, so $\sqrt{R/g}$ could have been used to scale time:

$$t^* := t/_{\sqrt{R/g}} = t\sqrt{\frac{g}{R}}.$$

Remark. Explicitly noting dependence on variables vs. parameters, the solution to
$$\begin{cases} d^2x/dt^2 = -gR/(x+R)^2 \\ x(0)=0, \quad \left.\frac{dx}{dt}\right|_{t=0} = V \end{cases}$$
is of the form
$$x = x(t\,;\, g, R, V),$$

whereas the sol'n to
$$\begin{cases} \varepsilon \frac{d^2x^*}{dt^{*2}} = -\frac{1}{(x^*+1)^2} \\ x(0)=0, \quad \left.\frac{dx^*}{dt^*}\right|_{t^*=0} = 1 \end{cases}$$
is of the form
$$x^* = x^*(t^*\,;\,\varepsilon), \quad \varepsilon = -\frac{V^2}{Rg}$$

Remark. In a properly nondimensionalized problem, all variables and parameters appear in DIMENSIONLESS GROUPS. (like $\varepsilon = \frac{-V^2}{Rg}$)

Moreover ... **ANY** complete/meaningful equation implies an equation in which all variables & parameters appear in dimensionless combinations/groups!

This last stmt. is called the BUCKINGHAM PI THEOREM.

Buckingham π Theorem. If we have a physically meaningful equation such as

$$f(q_1, q_2, \cdots, q_m) = 0,$$

where the q_i are dimensional variables/parameters (note: there are m many), and where those variables are expressed in terms of k many fundamental physical dimensions/units, then the eq'n can be restated as

$$F(\pi_1, \pi_2, \ldots, \pi_p) = 0, \quad \text{where } p = m-k \text{ and}$$

where the π_i are dimensionless combinations of the q_i:

$$\forall i \in [1,p] \cap \mathbb{N}, \quad \exists\, a_1, a_2, \ldots, a_m \in \mathbb{Z} \text{ s.t. } \pi_i = q_1^{a_1} q_2^{a_2} q_3^{a_3} \cdots q_m^{a_m}.$$

Aside for those who've taken/are taking linear algebra:

Consider the set V of all possible fundamental and derived physical units.

Suppose u and v are two such units; define

$u \oplus v$ to be the "multiplication" of the physical units (e.g., $kg \oplus m := kg \cdot m$).

Suppose $x \in \mathbb{Q}$, and define $x \odot u$ to be u^x (e.g., $3 \odot kg := kg^3$).

It can be proven that V, with $\oplus$ and $\odot$ as defined, is a VECTOR SPACE over $\mathbb{Q}$!

For the equation $f(q_1, q_2, \cdots, q_n) = 0$, set up a matrix M whose rows are the <u>fundamental</u> dimensions and whose columns are the variables:

$$
M = \begin{array}{cc}
\left[\begin{array}{cccc}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{k1} & m_{k2} & \cdots & m_{kn}
\end{array}\right] & \begin{array}{l} \leftarrow \text{dim. } 1 \\ \leftarrow \text{dim. } 2 \\ \quad \vdots \\ \leftarrow \text{dim. } k \end{array} \\
\begin{array}{cccc} \uparrow & \uparrow & \cdots & \uparrow \\ q_1 & q_2 & & q_n \end{array} &
\end{array}
$$

where m_{ij} is the power of the i^{th} fund'l dimension in the j^{th} variable/parameter.

<u>Example</u>:

$[x] = \mathcal{L}$ $\qquad$ $[v] = \mathcal{L}\mathcal{T}^{-1}$

$[t] = \mathcal{T}$ $\qquad$ $[g] = \mathcal{L}\mathcal{T}^{-2}$

$[R] = \mathcal{L}$ $\qquad$ <u>Fund'l units</u>: $\mathcal{L}$, $\mathcal{T}$

so

$$
M = \begin{array}{ccccc|c}
x & t & R & v & g & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\end{array}
$$

$$
M = \left[\begin{array}{ccccc}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & -1 & -2
\end{array}\right] \begin{array}{l} \leftarrow \mathcal{L} \\ \leftarrow \mathcal{T} \end{array}.
$$

The Buckingham Π Theorem states that this matrix M (called the <u>dimensional matrix</u>) has rank k ~~and nullity p~~ — and the <u>rank-nullity theorem</u> says that the nullity, p, is equal to $n-k$. (i.e., <u>$n = k+p$</u>).

Idea: Even if we don't know the equation — if we only know the parameters & variables and their dimensions — we can find the dimensionless combinations / groups π_i, which may reveal something fundamental about the problem and its solution!

Example / Illustration: PROJECTILE PROBLEM:

parameters are V, g, and R (disregard variables for now), and

$$[V] = \mathcal{L}\mathcal{T}^{-1}, \quad [g] = \mathcal{L}\mathcal{T}^{-2}, \quad [R] = \mathcal{L}.$$

Suppose $\pi = V^{a_1} g^{a_2} R^{a_3}$.

Then
$$[\pi] = [V]^{a_1}[g]^{a_2}[R]^{a_3}$$
$$= (\mathcal{L}\mathcal{T}^{-1})^{a_1}(\mathcal{L}\mathcal{T}^{-2})^{a_2}(\mathcal{L})^{a_3}$$
$$= \mathcal{L}^{a_1}\mathcal{T}^{-a_1}\mathcal{L}^{a_2}\mathcal{T}^{-2a_2}\mathcal{L}^{a_3}$$
$$= \mathcal{L}^{\underline{(a_1+a_2+a_3)}}\mathcal{T}^{\underline{(-a_1-2a_2)}}.$$

If we want $[\pi] = 1$, then we should have

$$a_1 + a_2 + a_3 = 0 \quad \text{and} \quad -a_1 - 2a_2 = 0.$$

Solve the system $\begin{cases} a_1 + a_2 + a_3 = 0 \\ -a_1 - 2a_2 = 0 \end{cases}$ however you like...

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & -2 & 0 & 0 \end{array}\right] \sim \begin{array}{c} R1 \\ R2+R1 \end{array} \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]}_{REF} \sim \begin{array}{c} R1+R2 \\ -R2 \end{array} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]}_{RREF}.$$

a_3 is the free parameter (could call a_1 or a_2 free, too – the point is that one of the variables is free), and the others are:

$$a_1 = -2a_3$$
$$a_2 = a_3 .$$

Check:
$$a_1 + a_2 + a_3 = (-2a_3) + (a_3) + a_3 = (-2+1+1)a_3 = 0 \checkmark$$
$$-a_1 - 2a_2 = -(-2a_3) - 2(a_3) = (2-2)a_3 = 0 \checkmark$$

So,

$$\Pi = V^{a_1} g^{a_2} R^{a_3}$$
$$= V^{-2a_3} g^{a_3} R^{a_3}$$
$$= (V^{-2} g R)^{a_3}$$
$$= \left(\frac{gR}{V^2}\right)^{a_3}.$$

Let $a_3 = -1$: $\Pi = \frac{V^2}{gR}$

Ships — want t/know power P (work per unit time) req. to keep ship length L moving straight @ speed u.

$$P = f(u, L, g, \rho, \nu), \quad \text{where } g: \text{grav'l}$$

accel., ρ: density, ν: kinematic viscosity.

1: $[P] = \mathcal{M}\mathcal{L}^2\mathcal{T}^{-3}$

2: $[L] = \mathcal{L}$

3: $[u] = \mathcal{L}\mathcal{T}^{-1}$

4: $[g] = \mathcal{L}\mathcal{T}^{-2}$

5: $[\rho] = \mathcal{M}\mathcal{L}^{-3}$

6: $[\nu] = \mathcal{L}^2\mathcal{T}^{-1}$

Assume $\Pi = P^{a_1} L^{a_2} u^{a_3} g^{a_4} \rho^{a_5} \nu^{a_6}$

$$[\Pi] = \mathcal{M}^{a_1}\mathcal{L}^{2a_1}\mathcal{T}^{-3a_1}\mathcal{L}^{a_2}\mathcal{L}^{a_3}\mathcal{T}^{-a_3}\mathcal{L}^{a_4}\mathcal{T}^{-2a_4}\mathcal{M}^{a_5}\mathcal{L}^{-3a_5}\mathcal{L}^{2a_6}\mathcal{T}^{-a_6}$$

$$= \mathcal{M}^{(a_1+a_5)}\mathcal{L}^{2a_1+a_2+a_3+a_4-3a_5+2a_6}\mathcal{T}^{\underbrace{-3a_1-a_3-2a_4-a_6}_{=0}}$$

$$\begin{cases} a_1 + a_5 = 0 \\ 2a_1 + a_2 + a_3 + a_4 - 3a_5 + 2a_6 = 0 \\ -3a_1 - a_3 - 2a_4 - a_6 = 0 \end{cases}$$

$$\Pi = M^{a_1+a_5} L^{2a_1+a_2+a_3+a_4-3a_5+2a_6} T^{-3a_1-a_3-2a_4-a_6}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & -3 & 2 \\ -3 & 0 & -1 & -2 & 0 & -1 \end{bmatrix} \sim \begin{array}{c} R1 \\ R2-2R1 \\ R3+3R1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -5 & 2 \\ 0 & 0 & -1 & -2 & 3 & -1 \end{bmatrix} \sim$$

$$\begin{array}{c} R1 \\ R2+R3 \\ -R3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{bmatrix}$$

$$a_1 = -a_5$$

$$a_2 = a_4 + 2a_5 - a_6$$

$$a_3 = -2a_4 + 3a_5 - a_6$$

$$P^{-a_5} L^{a_4+2a_5-a_6} u^{-2a_4+3a_5-a_6} g^{a_4} \rho^{a_5} \nu^{a_6}$$

$$\left(\frac{\rho u^3 L^2}{P}\right)^{a_5} \underbrace{\left(\frac{gL}{u^2}\right)^{a_4}} \underbrace{\left(\frac{\nu}{Lu}\right)^{a_6}}$$

when $a_4 = -\frac{1}{2}$, this qty. is $\boxed{\frac{u}{\sqrt{gL}}}$, the Froude number associated w/wave resistance

when $a_6 = -1$, the qty. is $\boxed{\frac{Lu}{\nu}}$, the Reynolds number, assoc. w/viscous resistance

$$\boxed{\frac{\rho u^3 L^2}{P} = F(Re, Fr)}$$