Lecture 1 : Sept. 13.

Announcements / Assignments.

- Hand in journals today (get them back on Thursday)
- There is a "When 2 Meet" on Canvas for deciding about a day/time especially for the "demos" – please fill out by end of day on Thursday.
- Questions on "administrative" issues? (I hope, if there are questions about Week 1 content, they'll be answered in today's lecture...

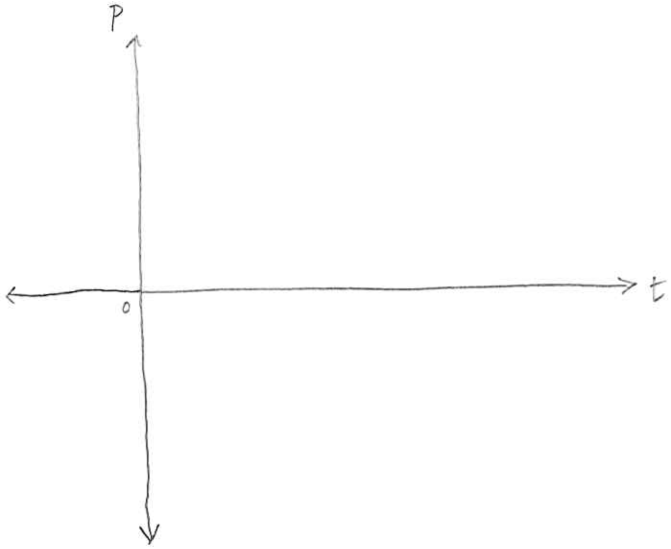
Today / "Week" 1.

- What is a differential eq'n?
- Classification
  - Order
  - Linearity
  - Homogeneity
  - Boundary/initial cond'ns
  - Ordinary/partial
- What is a sol'n to a differential eq'n?
- A bit abt. mathematical modelling.

Example (Torricelli's Law): "The time rate of change of the volume V of water draining from a tank is proportional to the square root of the depth, y, of water in the tank:

Example (Population Dynamics): The time rate of change of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population:

Can we think of a function that satisfies this equation?



Example (bacteria): Suppose $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at $t = 0$ hours was 1,000, and that the population doubled after one hour.

Classification of differential eq'ns.

(1) Ordinary vs. Partial.

If the unknown function depends on a single independent variable, then only "ordinary" derivatives appear in the differential eq'n, and it is said to be an ordinary diff'l eq'n, or ODE.

All DE's we've seen so far today have been ODEs.

Another ODE is

$$L \frac{d^2 Q(t)}{dt^2} + R \frac{d Q(t)}{dt} + \frac{1}{C} Q(t) = E(t),$$

for the charge $Q(t)$ on a capacitor in a circuit with capacitance C (const.), resistance R (const.) and inductance L (also const.).

Eq'ns where unknown function has ≥ 2 indep. variables have partial derivatives and are called PDEs.

Example (Wave Eq'n): $$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}.$$

This class is about ODE only.

(2) Systems of DE.

This classificat'n depends on the number of unknown functions involved. If we want to find two or more functions, then a system of DEs is required.

Example (Lotka-Volterra Eq'ns for population modelling)

$$\begin{cases} \dfrac{dx}{dt} = ax - \alpha xy \\[2ex] \dfrac{dy}{dt} = -cy + \gamma xy \end{cases}$$

where $x(t)$ is the population of prey, $y(t)$ the pop. of predators. Constants depend on species.

(3) Order.

The order of a DE is the order of the highest derivative that appears in the equation.

Examples.

$$y''' + 2e^t y'' + yy' = 4 \quad \text{is } 3^{rd} \text{ order}$$

$$y'' - y'' + (y')^2 + ty' + 4y = 0 \quad \text{is } 2^{nd} \text{ order.}$$

(3) Order, ct'd.

Generally, $F(t, y, y', \ldots, y^{(n)}) = 0$ is a DE of n^{th} order.

Assuming we can always solve for the highest derivative, can write the above eq'n as

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}).$$

This sometimes avoids ambiguity —

Example. $(y')^2 + ty' + 4y = 0$ is of the 1st form, but it leads to the two eq'ns: $y' = \frac{-t + \sqrt{t^2 - 16y}}{2}$ or $y' = \frac{-t - \sqrt{t^2 - 16y}}{2}$. Just writing the latter two eq'ns (or either one of them) would have avoided ambiguity.

④ Linearity / Nonlinearity.

A crucial classification; the ODE $F(t, y, y', \dots, y^{(n)}) = 0$ is called <u>linear</u> if F is a linear function of $y, y', \dots, y^{(n)}$.

The general, n^{th} order, Linear ODE is:

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \dots + a_n(t)\, y = g(t),$$

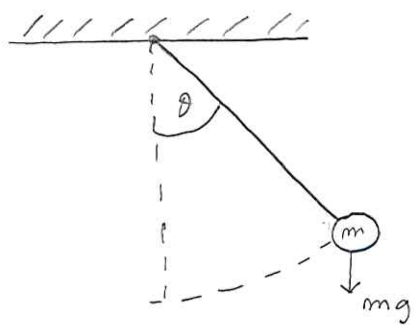
where the $a_i(t)$ are functions of t only.

An ODE that is not of this form is called nonlinear.

<u>example.</u> $y''' + 2e^t y'' + yy' = t^4$

What makes this eqn nonlinear?

<u>Example.</u>
Oscillating pendulum:



$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

What makes this eq'n nonlinear?

(6) Existence / Uniqueness / Solvability.

- Not every ODE has a solution! Those that do satisfy the "existence" criterion.

  There are some theorems for determining existence of solutions before putting in the effort to find solutions.

  Offers a kind of "first reality check" for math. modellers: If your equation or system models real-world phenomena, then because these phenomena actually occurred in real life, the eq'n(s) should have solutions!

- Assuming the ODE has at least one sol'n, next question is of uniqueness: could there be more than one solution? What additional conditions must be met in order to specify only one solution? Sometimes this means initial or boundary conditions need to be formulated.

- Can we actually find sol'ns? Is the eq'n solvable? Do we need to approximate sol'ns w/a computer?

⑦ Initial / Boundary value problem.

An initial value problem (IVP) is an ODE together with an initial condition of the form $y(0) = y_0$.

Arises in modelling real-life phenomena.

We learned last time that, formally, a differential equation is just an equation relating an unknown function to one or more of its derivatives.

To the mathematical modeller, they are more than that:

Derivative is the rate of change of a function with respect to its independent variable...

... and the physical universe revolves around change, e.g.,

- Population dynamics
- Physical motion — solids, fluids
- Electrical circuits
- Heat dissipation

Example (Newton's Law of Cooling): "The time rate of change of the temperature T is proportional to the difference between T and the surrounding (ambient) temperature A.

"In math":

(4) Linearity, ct'd.

There is well-developed theory describing how to solve linear ODEs, but in contrast, nonlinear ODEs are more complicated.

Sometimes, though, can linearize – i.e., approximate the nonlinear coefficient.

Example. If θ is small, then $\sin\theta \sim \theta$. So the pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

Linearization is an extremely valuable tool for handling nonlinearity, which arises frequently in physical descriptions.

(5) Solutions.

A solution of the ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is a function $\varphi(t)$ whose derivatives $\varphi', \varphi'', \ldots, \varphi^{(n)}$ exist, and for which $F(t, \varphi, \varphi', \ldots, \varphi^{(n)}) = 0$.

Usually not easy to solve ODE; easier to verify sol'ns by plugging in.