Oct. 27.

Recall: To solve the homogeneous, n^{th} order linear ODE with const. coeffs.

$$a_m \frac{d^m y}{dx^m} + a_{m-1}\frac{d^{m-1}y}{dx^{m-1}} + \cdots + a_3 \frac{d^3y}{dx^3} + a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

We assume sol'ns of the form $y(x) = e^{px}$, so that when substituting, $\frac{d^k y}{dx^k} = p^k e^{px}$ and the ODE becomes

$$a_m p^m e^{px} + a_{m-1} p^{m-1} e^{px} + \cdots + a_2 p^2 e^{px} + a_1 p e^{px} + a_0 e^{px} = 0$$

$$\underbrace{e^{px}}_{\neq 0}\underbrace{\left[a_m p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0\right]}_{\text{"characteristic eq'n"}} = 0$$

set to zero, solve for p. The resulting values of p (which exist due to the fund'l th'm of algebra) make $y = e^{px}$ a sol'n of the ODE.

(1) All roots of the characteristic eq'n are real and distinct. So

$$y(x) = c_1 e^{p_1 x} + c_2 e^{p_2 x} + \cdots + c_m e^{p_m x}$$

is the general sol'n to the ODE.

<u>Example</u>: Exercise 20 (p. 220), prob. 1

$$y'' + 2y' = 0.$$

Assume sol'ns of the form $y(x) = e^{px}$, so $y' = pe^{px}$, $y'' = p^2 e^{px}$.

The ODE becomes, upon substituting,

$$p^2 e^{px} + 2pe^{px} = 0, \quad \text{i.e.,}$$

$$e^{px}\left[p^2 + 2p\right] = 0.$$

The characteristic eq'n is $p^2 + 2p = 0$, i.e., $p(p+2) = 0$, which is solved by $p_1 = 0$ and by $p_2 = -2$.

Therefore, $y_1(x) := c_1 e^{0x} = c_1$ and $y_2(x) := c_2 e^{-2x}$ are both sol'ns of the ODE. If the functions 1 and e^{-2x} are lin. <u>in</u>dep., then the general sol'n to the ODE is

$$y(x) = c_1 + c_2 e^{-2x}.$$

To show linear independence of 1 and e^{-2x}, compute the Wronskian:

$$W\left(\{1, e^{-2x}\}\right) = \begin{vmatrix} 1 & e^{-2x} \\ 0 & -2e^{-2x} \end{vmatrix} = -2e^{-2x},$$

which is not identically zero on $\mathbb{R}$; therefore, 1 and e^{-2x} are, indeed, lin. indep. — so, $y(x) := c_1 + c_2 e^{-2x}$ is the gen. sol'n.

<u>Example (Prob. 1, Ex. 20, ct'd.)</u>

<u>Check</u>: $y(x) := c_1 + c_2 e^{-2x}$ solves $y'' + 2y' = 0$.

$y'(x) = -2c_2 e^{-2x}$

$y''(x) = 4c_2 e^{-2x}$, so the LHS of the ODE is:

$$y'' + 2y' = 4c_2 e^{-2x} + 2\left[-2c_2 e^{-2x}\right]$$

$$= 4c_2 e^{-2x} - 4c_2 e^{-2x} = 0. \quad \checkmark$$

<u>20C: Characteristic eq'n has repeated roots.</u>

For example, the ODE $y'' - 4y' + 4y = 0$ has the characteristic eq'n $p^2 - 4p + 4 = 0$, which is identical to the factored form $(p-2)^2 = 0$. This has roots $p_1 = p_2 = 2$. If we were to write the general sol'n of the ODE, we would have

this is just one sol'n to the ODE.

$$y(x) := c_1 e^{2x} + c_2 e^{2x} = (c_1 + c_2) e^{2x}.$$

Our existence + uniqueness thm guaranteed two l. ind. sol'ns to the 2nd-ord. eq'n. Need to find the other sol'n.

The assump'n tht. led to the characteristic eq'n was that all sol'ns of the ODE had the form $y(x) = e^{px}$. Let's, instead, assume sol'ns of the form $y(x) := u(x) e^{px}$.

20 C, ct'd.

For generality, assume the ODE in question has the form

$$y'' - 2ay' + a^2 y = 0,$$

whose characteristic eq'n is $p^2 - 2ap + a^2 = 0$, i.e., $(p-a)^2 = 0$.

We've assumed sol'ns of the form $y(x) := u(x)e^{px}$. So $y'(x) = u'(x)e^{px} + pu(x)e^{px}$, and $y''(x) = u''(x)e^{px} + pu'(x)e^{px} + p^2u(x)e^{px} + pu'(x)e^{px}$.

That is, $y''(x) = u''(x)e^{px} + 2pu'(x)e^{px} + p^2u(x)e^{px}$. The LHS of the ODE becomes

$$y'' - 2ay' + a^2y = \left[u''e^{px} + 2pu'e^{px} + p^2ue^{px}\right] - 2a\left[u'e^{px} + pue^{px}\right] + a^2ue^{px}$$

$$= e^{px}\left[u'' + 2pu' + p^2u - 2au' - 2apu + a^2u\right].$$

$$= e^{px}\left[u'' + u'(2p-2a) + u(p^2 + a^2 - 2ap)\right].$$

Notice: if $p=a$, the LHS of the ODE becomes

$$y'' - 2ay' + a^2y = e^{ax}\left[u'' + \underbrace{u'(2a-2a)}_{=0} + \underbrace{u(a^2+a^2-2a^2)}_{=0}\right]$$

$$= u''(x)e^{ax}.$$

To satisfy the ODE, wanted $y'' - 2ay' + a^2y = 0$, i.e., want $u''(x)\underbrace{e^{ax}}_{\neq 0} = 0$. So $u''(x) = 0$. That is, 20c, ct'd.

$u''(x) = 0$ implies $\underbrace{\int u''(x)\,dx}_{= u'(x) + c} = \underbrace{\int 0\,dx}_{=0}$, so $u'(x) = c_2$,

a constant. Integrate again: $\int u'(x)\,dx = \int c_2\,dx$, so

$u(x) = c_2 x + c_1$. Check: $u''(x) = \frac{d^2}{dx^2}\left[c_2 x + c_1\right] = 0$.

We've found that under the assump'n $y(x) = u(x)e^{px}$, $p = a$ and $u(x) = c_2 x + c_1$ ~~[illegible]~~ yield a solution of the ODE $y'' - 2ay' + a^2 y = 0$. That is,

$y(x) = (c_1 x + c_2)e^{ax}$ is the general sol'n of $y'' - 2ay' + a^2 y = 0$.

Our linearly indep. sol'ns are xe^{ax} and e^{ax}.

Check using Wronskian:

$$W(\{e^{ax}, xe^{ax}\}) = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & e^{ax} + axe^{ax} \end{vmatrix}$$

$$= e^{ax}\left(e^{ax} + axe^{ax}\right) - ae^{ax}\left(xe^{ax}\right)$$

$$= e^{2ax} + axe^{2ax} - axe^{2ax}$$

$$= e^{2ax},$$

which is nonzero for at least one $x \in \mathbb{R}$ (in fact, is nonzero for all of them), so $\{e^{ax}, xe^{ax}\}$ is linearly independent. So the general sol'n of $y'' - 2ay' + a^2 y = 0$ is, indeed, $y(x) = (c_1 + c_2 x)e^{ax}$.

20c, ct'd.

Check: $y(x) = c_1 e^{ax} + c_2 x e^{ax}$ has derivatives

$$y'(x) = c_1 a e^{ax} + c_2 e^{ax} + c_2 a x e^{ax}$$

$$y''(x) = c_1 a^2 e^{ax} + c_2 a e^{ax} + c_2 a e^{ax} + c_2 a^2 x e^{ax}$$

$$= (c_1 a^2 + 2c_2 a) e^{ax} + c_2 a^2 x e^{ax}.$$

So
$$y'' - 2ay' + a^2 y = \left[(c_1 a^2 + 2c_2 a) e^{ax} + c_2 a^2 x e^{ax}\right] - 2a\left[(c_1 a + c_2) e^{ax} + c_2 a x e^{ax}\right] + a^2\left[c_1 e^{ax} + c_2 x e^{ax}\right]$$

$$= e^{ax}\left[c_1 a^2 + 2c_2 a + c_2 a^2 x - 2c_1 a^2 - 2c_2 a - 2c_2 a^2 x + c_1 a^2 + c_2 a^2 x\right]$$

$$= e^{ax}\left[0\right] = 0 \text{, so ODE is satisfied.}$$

Exercise 20 (p. 220), Prob. 13

$\dfrac{d^4 y}{dx^4} = 0$. The characteristic eq'n is $p^4 = 0$, which has the four-times-repeated root $p = 0$. We can immediately write the general sol'n

$$y(x) = \left[c_3 x^3 + c_2 x^2 + c_1 x + c_0\right] e^{0x}$$

$y(x) = c_3 x^3 + c_2 x^2 + c_1 x + c_0$, whose 4th deriv. is zero, which satisfies the ODE. $\{1, x, x^2, x^3\}$ is linearly independent (check the Wronskian).

Example 20.43

$$\frac{d^4y}{dx^4} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0.$$

The characteristic eq'n is $p^4 - 3p^2 + 2p = 0$, so

$p(p^3 - 3p + 2) = 0$, so $p(p-1)(\underline{\quad\quad}) = 0$.

$$\begin{array}{r|l} & p^2 + p - 2 \\ \hline p-1 & p^3 - 3p + 2 \\ & -(p^3 - p^2) \\ \hline & p^2 - 3p + 2 \\ & -(p^2 - p) \\ \hline & -2p + 2 \\ & -(-2p + 2) \\ \hline & 0 \end{array}$$

That is,

$$\begin{aligned} p^4 - 3p^2 + 2p &= p(p-1)(p^2 + p - 2) \\ &= p(p-1)(p+2)(p-1) \\ &= p(p-1)^2(p+2), \end{aligned}$$

which is solved by $p_1 = 0$, $p_2 = p_3 = 1$, and $p_4 = -2$.

The general sol'n is:

$$\begin{aligned} y(x) &= c_1 e^{0x} + (c_2 + c_3 x)e^{1x} + c_4 e^{-2x} \\ &= c_1 + c_2 e^x + c_3 x e^x + c_4 e^{-2x}. \end{aligned}$$

Our 4 lin. indep. sol'ns to the 4^{th} order ODE are

$y_1(x) = 1$, $y_2(x) = e^x$, $y_3(x) = xe^x$, $y_4(x) = e^{-2x}$.