20D: Linear Homogeneous m^{th} order, Constant coeff. ODE, Characteristic eq'n has imaginary or complex roots.

$$a_m \frac{d^m y}{dx^m} + a_{m-1} \frac{d^{m-1} y}{dx^{m-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 .$$

Assume $y = e^{px}$; this leads to the characteristic eq'n

$$a_m p^m + a_{m-1} p^{m-1} + \cdots + a_2 p^2 + a_1 p + a_0 = 0 .$$

If all coefficients a_i are real, any complex sol'ns come in conjugate pairs: $\alpha + i\beta$ is a root implies $\alpha - i\beta$ is also a root.

Assuming $\alpha + i\beta$ and $\alpha - i\beta$ solve the char. eq'n, sol'ns to the ODE take the form

$$\begin{aligned} y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \\ &= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x} \\ &= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right). \end{aligned}$$

A short digression on Euler's Theorem...

Recall: The Taylor series of e^x, for $x \in \mathbb{R}$, is

$$e^x = \sum_{m=0}^{\infty} \frac{x^m \left(\frac{d^m}{dx^m} e^x \big|_{x=0}\right)}{m!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots$$

Big step: to define (yes, define) exponentiation of complex or imaginary numbers, we'd like to choose a convention that satisfies the Taylor expansions:

$$e^{i\theta} = 1 + \theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

Recall: $i := \sqrt{-1}$, so

$$i^2 = -1$$
$$i^3 = i^2 i = -i$$
$$i^4 = (i^2)^2 = 1$$
$$i^5 = i^4 i = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$
$$\vdots$$

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \dots$$

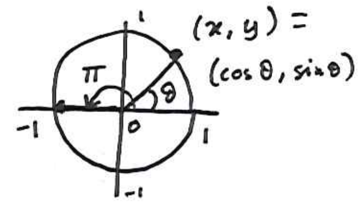
$$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \frac{i\theta^9}{9!} - \dots$$

$$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right] + i\left[\theta - \frac{\theta^2}{2!} + \frac{\theta^3}{3!} - \dots\right]$$

$$= \cos\theta + i\sin\theta.$$

So, we *define* exponentiation of imaginary numbers as:

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$$



For example,

$$e^{\pi i} = \cos\pi + i\sin\pi$$

$$e^{\pi i} = -1 + i(0)$$

$$e^{\pi i} = -1. \quad \text{Euler's formula.}$$

Relates the 4 most important constants in mathematics: e, π, i, and -1.

When the char. eq'n of a linear, c.c., homog., n^{th} order ODE had a cplx root $\alpha + i\beta$, it also had the root $\alpha - i\beta$, and so a sol'n would be

$$y = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left[c_1\left(\cos(\beta x) + i\sin(\beta x)\right) + c_2\left(\cos(-\beta x) + i\sin(-\beta x)\right)\right]$$

$$= e^{\alpha x}\left[c_1\cos(\beta x) + c_1 i\sin(\beta x) + c_2\cos(\beta x) - c_2 i\sin(\beta x)\right]$$

$$= e^{\alpha x}\left[(c_1 + c_2)\cos(\beta x) + i(c_1 - c_2)\sin\beta x\right]$$

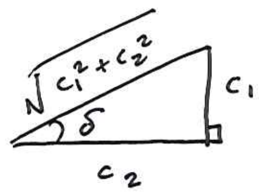
$$= e^{\alpha x}\left[\bar{c}_1\cos(\beta x) + i\,\bar{c}_2\sin(\beta x)\right], \quad \bar{c}_1 := c_1 + c_2, \quad \bar{c}_2 := c_1 - c_2$$

$$\boxed{y = e^{\alpha x}\left[c_1\cos(\beta x) + i\,c_2\sin(\beta x)\right],} \quad c_1 := \bar{c}_1, \quad c_2 := \bar{c}_2$$

$$\boxed{c_1 \cos\beta x + c_2 \sin\beta x = \sqrt{c_1^2 + c_2^2}\ \sin(\beta x + \delta)}, \quad \sin\delta = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$$

$$\cos\delta = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$$

$$\boxed{c_1 \cos\beta x + c_2 \sin\beta x = \sqrt{c_1^2+c_2^2}\ \cos(\beta x - \delta)}$$



The sol'ns to the ODE become

$$\left.\begin{array}{l} \boxed{y = e^{\alpha x}\sqrt{c_1^2+c_2^2}\ \sin(\beta x + \delta)} \\ \boxed{y = e^{\alpha x}\sqrt{c_1^2+c_2^2}\ \cos(\beta x - \delta)} \\ \boxed{y = e^{\alpha x}(c_1 \cos\beta x + i c_2 \sin\beta x)} \end{array}\right\}$$

3 total ways of writing the sol'n corresponding to $p = \alpha \pm i\beta$.

Example 20.61, p. 219 : Solve $y'' - 2y' + 2y = 0$.

$p^2 - 2p + 2 = 0$ is the char. eq'n obtained by assuming sol'ns of the form $y = e^{px}$.

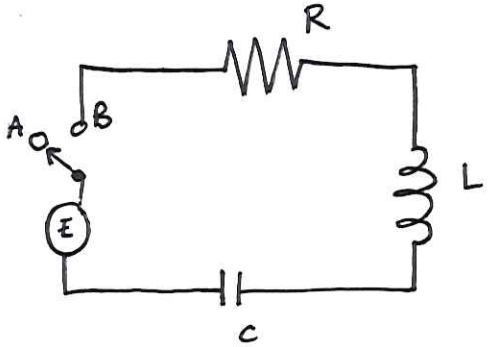
By the quadr. formula, the roots are $p = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$,

i.e., $p = \frac{2}{2} \pm \frac{1}{2}\sqrt{-4} = 1 \pm i$. So $\alpha = \beta = 1$.

Then the general sol'n of the ODE is:

$$\boxed{y = e^{x}(c_1 \cos x + i c_2 \sin x)}, \text{ or } y = e^{x}\cos(x - \delta) \quad y = e^{x}\sin(x + \delta).$$

30 A.



Voltage drop across a resistor $= Ri$

— " ——— an inductor $= L\dfrac{di}{dt}$

— " ——— a capacitor $= \dfrac{1}{c}\, q$

where R is the resistance in ohms Ω ; L is the inductance in henrys H, C is capacitance in farads F, q is the charge in coulombs c, i is the current in amperes A.

Think of velocity as the derivative of pos'n — analogue concept is $\boxed{i = \dfrac{dq}{dt}}$

Kirchhoff's 2nd law : emf of E $= \sum$ voltage drops in the closed circuit.

For our RLC circuit,

$$E(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\, q$$

$$E(t) = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{c}\, q$$

$$\boxed{E(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}\, q}$$