21AB : Undetermined Coeffs.

Linear, n^{th} order constant coefficient, non-homogeneous ODE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y + a_0 = Q(x)$$

Recall: In L19, a theorem: $y(x) = \underbrace{y_h(x)}_{\text{homog.}} + \underbrace{y_p(x)}_{\text{particular sol'n}}$ is the general sol'n to the linear n^{th} ord., const. coeff., inhomog. ODE.

To find y_p, we write (first) a "generic" version of $Q(x)$.

Example (1) Find a particular sol'n of

$$y'' + 3y' + 4y = \underbrace{3x + 2}_{\text{"forcing term" or "right-hand side"}}$$

$Q(x) = 3x + 2$. Our guess for $y_p(x)$ is the "generic" version $y_p(x) = Ax + B$. To substitute into the ODE, find $y_p'(x) = A$ and $y_p''(x) = 0$. Substituting:

$$\begin{aligned} y_p'' + 3y_p' + 4y_p &= 0 + 3A + 4(Ax + B) \\ &= 4Ax + (4B + 3A) = Q(x) = 3x + 2. \end{aligned}$$

So $4A = 3$, and $4B + 3A = 2$. That is, $A = 3/4$, and $B = \frac{2 - 9/4}{4} = \frac{1}{2} - \frac{9}{16} = -\frac{1}{16}$. So $y_p(x) = \frac{3}{4}x - \frac{1}{16}$.

Example ① ct'd.

The homog. sol'n $y_h(x)$ solves

$$y'' + 3y' + 4y = 0.$$

Guess: $y_h(x) = e^{px}$, some p. We obtain the characteristic eq'n

$$p^2 + 3p + 4 = 0$$

$$p = \frac{3^2 - \sqrt{3^2 - 4(4)(1)}}{2(1)} = \frac{9}{2} \pm \frac{1}{2}\sqrt{9-16}$$

$$= \frac{9}{2} \pm \frac{1}{2}\sqrt{-7}$$

$$= \boxed{\frac{9}{2}} \pm \boxed{\frac{i}{2}\sqrt{7}}$$

Know that. $y_h(x) = e^{\frac{9}{2}x}\left(c_1 \cos\left(\frac{x}{2}\sqrt{7}\right) + c_2 \sin\left(\frac{x}{2}\sqrt{7}\right)\right).$

So the gen. sol'n to $y'' + 3y' + 4y = 3x + 2$ is

$$y(x) = y_p(x) + y_h(x)$$

$$\boxed{y(x) = \frac{3}{4}x - \frac{1}{16} + e^{\frac{9}{2}x}\left(c_1 \cos\left(\frac{x}{2}\sqrt{7}\right) + c_2 \sin\left(\frac{x}{2}\sqrt{7}\right)\right)}$$

Example ② $y'' - 4y = 2e^{3x}$. Here, $Q(x) = 2e^{3x}$,

so a reasonable guess for $y_p(x)$ is $y_p(x) = Ae^{3x}$. So

$y_p'(x) = 3Ae^{3x}$, and $y_p''(x) = 9Ae^{3x}$. Substituting into the ODE,

find $y_p'' - 4y_p = 9Ae^{3x} - 4Ae^{3x} = 5Ae^{3x}$. Set this equal to $Q(x)$,

so $5Ae^{3x} = 2e^{3x}$, which implies $A = \frac{2}{5}$. So $y_p(x) = \frac{2}{5}e^{3x}$.

To find the homog. sol'n $y_h(x)$, we solve $y'' - 4y = 0$. We guess for $y_h(x)$ is $y_h(x) = e^{px}$, whereby we obtain the characteristic eq'n $p^2 - 4 = 0$. So $p = \pm 2$, i.e., $p_1 = -2$ and $p_2 = 2$, so

$$y_h(x) = c_1 e^{-2x} + c_2 e^{2x}.$$

The general sol'n to $y'' - 4y = 2e^{3x}$ is therefore

$$\boxed{y(x) = \frac{2}{5}e^{3x} + c_1 e^{-2x} + c_2 e^{2x}.}$$

Example ③ $\quad 3y'' + y' - 2y = 2\cos(x)$.

For this example, $Q(x) = 2\cos(x)$, so guess $y_p(x) = A\sin(x) + B\cos(x)$

$y_p'(x) = \frac{d}{dx}[A\sin x + B\cos x] = A\cos x - B\sin x$, and

$y_p''(x) = \frac{d}{dx}[A\cos x - B\sin x] = -A\sin x - B\cos x$. Substituting into LHS of ODE,

we obtain $\quad 3[-A\sin x - B\cos x] + [A\cos x - B\sin x] - 2[A\sin x + B\cos x] =$

$$= (-3A - B - 2A)\sin x + (-3B + A - 2B)\cos x$$

$$= (-5A - B)\sin x + (A - 5B)\cos x.$$

Setting this equal to $Q(x)$, we obtain

$$2\cos x = (-5A - B)\sin x + (A - 5B)\cos x.$$

This implies

$$2 = A - 5B \quad \text{and} \quad 0 = -5A - B.$$

Solve the linear system $\begin{cases} A - 5B = 2 \\ -5A - B = 0 \end{cases}$.

Could substitute: $B = -5A$ implies $A - 5(-5A) = 2 \Rightarrow +26A = 2$
$\Rightarrow A = +1/13$

So $B = -5/13$.

Could solve linear sys.:

$$\begin{array}{r} 5(A - 5B) = 5(2) \\ +(-5A - B = 0) \\ \hline 0A - 26B = 10 \end{array} \Rightarrow B = -5/13$$

$$A = 2 + 5B = \tfrac{1}{13}$$

Ex. 3 ct'd.

$$\begin{cases} A-5B=2 \\ -5A-B=0 \end{cases}$$

$A-5B=2$ implies $A=2+5B$; subst. into $-5A-B=0$ yields

$-5(2+5B)-B=0$ or $-10-25B-B=0$ or $-26B=10$ or $B=\frac{-5}{13}$,

so $A = 2+5\left(\frac{-5}{13}\right) = 2-\frac{25}{13} = \frac{26-25}{13} = \frac{1}{13}$.

We obtain $\begin{cases} A=1/13 \\ B=-5/13 \end{cases}$. Therefore, $y_p(x) = \frac{1}{13}\sin x - \frac{5}{13}\cos x$.

The homog. sol'n satisfies $3y''+y'-2y=0$.

Guess: $y_h(x)=e^{px}$, obtain the char. eq'n $3p^2+p-2=0$

That is $\underbrace{(3p-2)(p+1)}_{\substack{3p^2+3p-2p-2 \\ =3p^2+p-2}}=0$, solved by $p_1=-1$ and $p_2=\frac{2}{3}$.

(Note: could also have used quadr. formula: $p = \frac{-1\pm\sqrt{1^2-4(-2)(3)}}{2(3)}$

$$= \frac{-1\pm\sqrt{25}}{2(3)}$$

$$= \frac{-1\pm 5}{6}$$

$$p_1 = \frac{-1-5}{6} = \frac{-6}{6} = -1, \quad p_2 = \frac{-1+5}{6} = \frac{4}{6} = \frac{2}{3}$$

Distinct real roots, so $y_h(x) = c_1e^{-x}+c_2e^{\frac{2}{3}x}$.

Ex ③ ct'd

The general sol'n of $3y''+y'-2y=2\cos(x)$ is

$$y(x) = y_p(x) + y_h(x)$$

$$\boxed{y(x) = \frac{1}{13}\sin x - \frac{5}{13}\cos x + c_1 e^{-x} + c_2 e^{\frac{2}{3}x}.}$$

Example ④ ct'd.

$y'' - 4y = 2e^{2x}$. $Q(x) = 2e^{2x}$, so we guess

$y_p(x) = Ae^{2x}$, so $y_p''(x) = 4Ae^{2x}$. Substituting into the LHS of the ODE, we obtain

$$y_p'' - 4y_p = 4Ae^{2x} - 4(Ae^{2x})$$
$$= 0.$$

This method degenerated because e^{2x} is a sol'n of the homog. eq'n, and also appears in the forcing term $Q(x)$.

Second try: $y_p(x) = Axe^{2x}$. Then $y_p'(x) = Ae^{2x} + 2Axe^{2x}$,

and $y_p''(x) = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$. Substituting into LHS of ODE,

$y_p'' - 4y_p = (4Ae^{2x} + 4Axe^{2x}) - 4(Axe^{2x}) = 4Ae^{2x}$. Setting this equal to $Q(x)$ yields $4Ae^{2x} = 2e^{2x}$, so $A = \frac{1}{2}$, and

$y_p(x) = \frac{1}{2}xe^{2x}$.

Recall that the homog. sol'n $y_h(x)$ solves $y'' - 4y = 0$, and from Example ②, we found $y_h(x) = c_1e^{-2x} + c_2e^{2x}$. So the gen. sol'n. is $\boxed{y(x) = \frac{1}{2}xe^{2x} + c_1e^{-2x} + c_2e^{2x} = c_1e^{-2x} + (\frac{1}{2}x + c_2)e^{2x}}$

The method of undetermined coefficients applies when the forcing term (source term, RHS) is a linear combinat'n of (finite) products of functions of the following 3 types:

(1) polynomial in x

(2) exponential e^{rx}

(3) $\cos(kx)$ or $\sin(kx)$

$\boxed{1}$ If no term appearing in $Q(x)$ or its derivatives satisfies the homogeneous ODE, then guess $y_p(x)$ is a lin. comb. of all the lin. indep. terms in $Q(x)$. Then solve for the coeffs.

$\boxed{2}$ We'll formalize on Tues., but see Example 4.