Nov. 8, 2016: Finishing Undet. Coeff., Starting Variat'n Params.

Example (of where Rule 1 fails).

Suppose we want a particular sol'n of

$$y''' + y'' = 3e^x + 4x^2.$$

Forcing term (RHS, source term) is $3e^x + 4x^2 =: Q(x)$,

so $Q'(x) = 3e^x + 8x$, $Q''(x) = 8 + 3e^x$. So the derivatives of $Q(x)$ look like the exponential plus a generic 2^{nd} degree polynomial.

We guess a particular sol'n:

$$y_p(x) := Ae^x + Bx^2 + Cx + D.$$

(Note: Rule 1 would have us guess $Ae^x + Bx^2$.)

$$\left.\begin{aligned} y_p'(x) &= Ae^x + 2Bx + C \\ y_p''(x) &= Ae^x + 2B \\ y_p'''(x) &= Ae^x \end{aligned}\right\} \quad 2Ae^x + 2B = 3e^x + 4x^2.$$

This guess was not a good one, but — it did reveal what other kinds of terms to include in our guess.

Second guess: $y_p(x) := Ae^x + Bx^4 + Cx^3 + Dx^2 \underbrace{+Ex+F}_{\text{not included, as } 2^{nd} \,\&\, 3^{rd} \text{ derivs. will vanish.}}$

$$y_p'(x) = Ae^x + 4Bx^3 + 3Cx^2 + 2Dx$$

$$y_p''(x) = Ae^x + 12Bx^2 + 6Cx + 2D$$

$$y_p'''(x) = Ae^x + 24Bx + 6C.$$

Substitute into LHS of ODE:

Ex (1) cont'd

$$y_p''' + y_p'' = 2Ae^x + 12Bx^2 + (24B+6C)x + (6C+2D)$$
$$= 2Ae^x + 12Bx^2 + 6(4B+C)x + 2(3C+D).$$
$$= 3e^x + 4x^2.$$

Because $\{e^x, x^2\}$ is lin. indep., must equate like terms:

$2A = 3$, $12B = 4$, $6(4B+C) = 0$, $2(3C+D) = 0$

$A = \frac{3}{2}$, $B = \frac{4}{12} = \frac{1}{3}$, $C = -4B = -\frac{4}{3}$, $D = -3C = 4$.

$$y_p(x) = \frac{3}{2}e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2.$$

Check: $y_p'(x) = \frac{3}{2}e^x + \frac{4}{3}x^3 - 4x^2 + 8x$

$y_p''(x) = \frac{3}{2}e^x + 4x^2 - 8x + 8$

$+ \; y_p'''(x) = \frac{3}{2}e^x + 8x - 8$

$y_p'''(x) + y_p''(x) = 3e^x + 4x^2 = Q(x).$

So the particular sol'n checks out.

$\boxed{\text{Rule 2: If } s \text{ is the smallest integer such that no term in } y_p \text{ duplicates any term in the homogeneous solution, then a guess for } y_p \text{ could be}}$

$$y_p(x) := x^s\left[(a_0 + a_1x + a_2x^2 + \dots + a_mx^m)e^{rx}\cos(\alpha x) + (b_0 + b_1x + b_2x^2 + \dots + b_mx^m)e^{rx}\sin(\alpha x)\right]$$

Ex ② $\quad y'' + 6y' + 13y = e^{-3x}\cos(2x)$.

Start by finding y_h, which solves $y'' + 6y' + 13y = 0$.

The characteristic eq'n is $p^2 + 6p + 13 = 0$, and the discriminant is $6^2 - 4(13)(1) = 36 - 52 = -16 < 0$, so our ~~sol'ns~~ roots are complex:

$$p = \frac{-6 \pm \sqrt{-16}}{2(1)} = -3 \pm 2i.$$

So $\quad y_h(x) = e^{-3x}\left(c_1\cos(2x) + c_2\sin(2x)\right)$.

Use Rule 2: the smallest positive integer s so that

$$y_p(x) := x^s\left[a_1 e^{-3x}\cos(2x) + b_1 e^{-3x}\sin(2x)\right]$$

doesn't duplicate terms in $y_h(x)$ above. This s-value is 1.

Therefore, our guess for $y_p(x)$ is

$$\boxed{y_p(x) := a_1 x e^{-3x}\cos(2x) + b_1 x e^{-3x}\sin(2x).}$$

Example 21.51, p. 230

Find a particular sol'n of $y'' - 3y' + 2y = \sin x$.

The homog. sol'n $y_h(x)$ solves $y'' - 3y' + 2y = 0$. The characteristic eq'n is $p^2 - 3p + 2 = 0$, with discriminant $\sqrt{(-3)^2 - 4(2)(1)} =$
$(p-2)(p-1) = 0$ $= \sqrt{9-8} = \sqrt{1} = 1 > 0$

So our roots are real, and they are:

$$p = \frac{3 \pm 1}{2} = 2, 1,$$

making $y_h(x) = ae^{2x} + be^{x}$. [$\underline{\text{Check}}$: $y_h'' - 3y_h' + 2y_h \stackrel{?}{=} 0$].

Our guess for $y_p(x)$ is $y_p(x) = c_1 \sin x + c_2 \cos x$.

$$y_p'(x) = c_1 \cos x - c_2 \sin x$$

$$y_p''(x) = -c_1 \sin x - c_2 \cos x = -y_p(x).$$

Substituting into LHS: $y_p'' - 3y_p' + 2y_p = -3y_p' + y_p$

$$= -3[c_1 \cos x - c_2 \sin x] + [c_1 \sin x + c_2 \cos x]$$

$$= (3c_2 + c_1)\sin x + (c_2 - 3c_1)\cos x.$$

Set $(3c_2 + c_1)\sin x + (c_2 - 3c_1)\cos x = \sin x$. Since $\{\sin x, \cos x\}$ is l.i., we must have $3c_2 + c_1 = \underline{1}$ and $c_2 - 3c_1 = 0$.

$$c_2 = 3c_1 \Rightarrow 3c_2 + c_1 = 3(3c_1) + c_1 = 10c_1 = 1 \Rightarrow c_1 = \frac{1}{10},$$

and $c_2 = \frac{3}{10}$. So $y_p(x) = \frac{1}{10}\sin(x) + \frac{3}{10}\cos(x)$.

[$\underline{\text{Check}}$: $y_p'' - 3y_p' + 2y_p \stackrel{?}{=} \sin x$] $y(x) = y_h(x) + y_p(x)$.

Example 21.51, cont'd.

Thus $y(x) = ae^{2x} + be^{x} + \frac{1}{10}\sin(x) + \frac{3}{10}\cos(x)$.

So, don't really have to use cplx. coefficients / variables.

(Pathological)
Example. $y'' + y = \tan(x)$.

Observe: $\frac{d}{dx}[\tan x] = \sec^2 x$, $\frac{d}{dx}[\sec^2 x] = 2\sec^2 x \tan x$,

$\frac{d}{dx}[2\sec^2 x \tan x] = 4\sec^2 x \tan^2 x + 2\sec^4 x, \ldots$

Derivatives don't ever cycle back – we have infinitely many linearly independent derivatives of $\tan(x)$, so there can be no finite trial guess for $y_p(x)$.

Undetermined coefficients won't work.

Different method: ① Find $y_h(x) = a_1 y_1(x) + a_2 y_2(x) + \cdots + a_m y_m(x)$

n^{th} ord.

② Guess that

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + \cdots + u_m(x) y_m(x)$$

③ Substitute into the ODE & find parameters $u_i(x)$.

§ we're solving the ODE $y'' + ay' + by = Q(x)$,

and § $y_h(x) = c_1 y_1 + c_2 y_2$.

By our method, seek u_1 & u_2 s.t. $\underline{y_p(x) := u_1 y_1 + u_2 y_2}$

solves $\underline{y'' + ay' + by} = \boxed{Q(x).}$

Well, $y_p'(x) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$.

Let's impose the stronger cond'n that $\boxed{u_1' y_1 + u_2' y_2 = 0.}$

If this is the case, then $\underline{y_p'(x) = u_1 y_1' + u_2 y_2'}$.

Then $y_p''(x) = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$.

Substitute into $\underline{\text{LHS}}$:

$$y_p'' + ay_p' + by_p = \left[u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''\right] + a\left[u_1 y_1' + u_2 y_2'\right] + b\left[u_1 y_1 + u_2 y_2\right]$$

$$= u_1\left[\underbrace{y_1'' + ay_1' + by_1}\right] + u_2\left[\underbrace{y_2'' + ay_2' + by_2}\right] + u_1' y_1' + u_2' y_2'.$$

Since y_1 & y_2 solve $y'' + ay' + by = 0$, we get:

$$= u_1[0] + u_2[0] + u_1' y_1' + u_2' y_2'$$
$$= u_1' y_1' + u_2' y_2'.$$

So $\boxed{u_1' y_1' + u_2' y_2' = Q(x).}$

Our two cond'ns were: $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = Q(x). \end{cases}$