Nov. 10 : Var. Params, SHM. 1

$$\left.\begin{array}{l} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = Q(x) \end{array}\right\} \text{ equiv. to } \underbrace{\underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{2\times 2} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}}_{2\times 1} = \underbrace{\begin{pmatrix} 0 \\ Q(x) \end{pmatrix}}_{2\times 1}.$$

The matrix system can be solved by multiplying both sides of the eq'n by the inverse of $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$.

The inverse of the 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\dfrac{1}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

So
$$\operatorname{inv}\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \frac{1}{\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}$$

$$= \frac{1}{W(\{y_1, y_2\})} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}.$$

Then
$$\underbrace{\frac{1}{W(\{y_1,y_2\})} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(\{y_1,y_2\})} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ Q(x) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(\{y_1,y_2\})} \underbrace{\underbrace{\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}}_{2\times 2} \begin{pmatrix} 0 \\ Q(x) \end{pmatrix}}_{2\times 1}$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(\{y_1,y_2\})} \begin{pmatrix} -y_2\, Q(x) \\ y_1\, Q(x) \end{pmatrix}.$$

So the system $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = 0 \end{cases}$ is solved by

$$u_1' = \frac{1}{W(\{y_1,y_2\})} \cdot (-y_2 Q), \quad u_2' = \frac{1}{W(\{y_1,y_2\})} (y_1 Q)$$

Final step: integrate

$$u_1(x) = \int \frac{-y_2(x)\,Q(x)\,dx}{W(\{y_1, y_2\})}, \qquad u_2(x) = \int \frac{y_1(x)\,Q(x)\,dx}{W(\{y_1, y_2\})}.$$

Substitute these into our guess for y_p:

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

$$y_p(x) = y_1 \int \frac{-y_2(x)\,Q(x)\,dx}{W(\{y_1, y_2\})} + y_2 \int \frac{y_1(x)\,Q(x)\,dx}{W(\{y_1, y_2\})}.$$

Example ①. $y'' + y = \tan x$

① Find $y_h(x)$ that solves $y'' + y = 0$.

The char. eq'n is $p^2 + 1 = 0$, solved by $p = \pm\sqrt{-1} = \pm i$,
$= 0 \pm 1i$

so the sol'n to the homog. eq'n is

$$y_h(x) = e^{0x}\left[c_1 \cos(1x) + c_2 \sin(1x)\right]$$

$$y_h(x) = c_1 \cos(x) + c_2 \sin(x).$$

② The linearly indep. basis fns. for y_h are $y_1(x) := \cos(x)$, $y_2(x) := \sin(x)$.

$$W(\{y_1, y_2\}) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

$$= \cos^2 x + \sin^2 x = 1.$$

(3) $$y_p(x) = y_1(x)\int \frac{-y_2(x)\,Q(x)\,dx}{W(\{y_1,y_2\})} + y_2(x)\int \frac{y_1(x)\,Q(x)\,dx}{W(\{y_1,y_2\})}$$

$$= \cos(x)\int \frac{-\sin x\tan x}{1}\,dx + \sin x\int \frac{\cos x\tan x}{1}\,dx$$

$$= \cos x\int -\frac{\sin^2 x}{\cos x}\,dx + \sin x\int \sin x\,dx$$

$\sin^2 x + \cos^2 x = 1$

$-\sin^2 x = \cos^2 x - 1$

$$= \cos x\int \frac{\cos^2 x - 1}{\cos x}\,dx + \sin x\int \sin x\,dx$$

$$= \cos x\left[\int \cos x\,dx - \int \frac{1}{\cos x}\,dx\right] + \sin x\int \sin x\,dx$$

$$= \cos x\left[\sin x - \underbrace{\int \frac{1}{\cos x}\,dx}\right] + \sin x(-\cos x)$$

Notice: $$\frac{d}{dx}\left[\underbrace{\frac{1+\sin x}{\cos x}}\right] = \frac{\cos x\frac{d}{dx}[1+\sin x] - (1+\sin x)\frac{d}{dx}[\cos x]}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - (1+\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1+\sin x}{\cos^2 x} = \frac{1}{\cos x}\left[\frac{1+\sin x}{\cos x}\right].$$

Let $v(x) := \dfrac{1+\sin x}{\cos x}$.

Then $\displaystyle\int \frac{1}{\cos x}\,dx = \int \frac{1}{\cos x}\,\frac{v(x)}{v(x)}\,dx$ (dv). But observe: $\dfrac{dv}{dx} = \dfrac{1}{\cos x}v$,

so $dv = \dfrac{v\,dx}{\cos x}$. So $\displaystyle\int \frac{1}{\cos x}\,dx = \int \frac{1}{v}\,dv$.

$$\int \frac{1}{\cos x}\, dx = \int \frac{1}{v}\, dv = \ln|v| + c$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + c$$

$$= \ln|\sec x + \tan x| + c.$$

So $$y_p(x) = \cos x\left[\sin x - \int \frac{1}{\cos x}\, dx\right] + \sin x\,(-\cos x + c_2)$$

$$= \cancel{\cos x \sin x} - \cancel{\cos(x) \sin x} - \cos x\left[\ln|\sec x + \tan x| + c_1\right] + c_2 \sin x$$

$$= \underbrace{-c_1 \cos x + c_2 \sin x} - \cos x \ln|\sec x + \tan x|$$

can take $c_1 = c_2 = 0$, and this is appropriate, because $\sin x$ & $\cos x$ each solves the homog. ODE.

So $$y_p(x) = -\cos x \ln|\sec x + \tan x|.$$

Therefore, the general sol'n to $y'' + y = \tan x$ is:

$$y(x) := y_h(x) + y_p(x)$$

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

Example 7, Ex. 22, p. 240

$$y'' + 2y' + y = \underbrace{x^2 e^{-x}}_{Q(x)}.$$

(1) Find y_h tht. solves $y'' + 2y' + y = 0$.

Char. eq'n is $p^2 + 2p + 1 = 0$, i.e., $(p+1)^2 = 0$.
So $p = -1$ is the (twice-)repeated root, and the sol'n

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

(2) $y_1(x) := e^{-x}$, $y_2(x) := xe^{-x}$.

$$W(\{y_1, y_2\}) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{pmatrix} =$$

$$= e^{-x}(e^{-x} - xe^{-x}) - (-e^{-x})(xe^{-x})$$

$$= e^{-2x} \cancel{- xe^{-x}} + \cancel{xe^{-2x}} = e^{-2x}.$$

(3) $y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, where:

$$u_1(x) = \int \frac{-y_2\, Q(x)}{W(\{y_1, y_2\})}\, dx, \quad u_2(x) = \int \frac{y_1(x)\, Q(x)\, dx}{W(\{y_1, y_2\})}.$$

$$u_1(x) = \int \frac{-xe^{-x}(x^2 e^{-x})}{e^{-2x}}\, dx = \int -x^3 \left(\frac{e^{-2x}}{e^{-2x}}\right) dx = \int -x^3\, dx = -\frac{1}{4}x^4$$

$$u_2(x) = \int \frac{e^{-x}(x^2 e^{-x})}{e^{-2x}}\, dx = \int x^2\, dx = \frac{1}{3}x^3.$$

So $y_p(x) = -\frac{1}{4}x^4 \underline{e^{-x}} + \frac{1}{3}x^3 \underline{xe^{-x}} = \left(\frac{1}{3} - \frac{1}{4}\right)x^4 e^{-x} = \frac{1}{12}x^4 e^{-x}$.

(4) The Gen. Sol'n: $y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{12} x^4 e^{-x}$.