Nov. 22 : Systems of ODE.

<u>Example 31.18</u>

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{t}{x^2}, & x \neq 0 \\[2ex] \dfrac{dy}{dt} = \dfrac{y}{t^2}, & t \neq 0 \end{cases}$$

Solve $\dfrac{dx}{dt} = \dfrac{t}{x^2}$. If $\dfrac{dx}{dt} = \dfrac{t}{x^2}$, then $x^2 \dfrac{dx}{dt} = t$.

Int. both sides: $\displaystyle\int x^2 \frac{dx}{dt}\, dt = \int t \, dt$

$$\int x^2 \, dx = \int t \, dt$$

$$\frac{1}{3} x^3 = \frac{1}{2} t^2 + \tilde{C}$$

$$\boxed{x = \left(\frac{3}{2} t^2 + c_1\right)^{1/3}} \qquad c_1 \doteq 3\tilde{C}.$$

Solve $\dfrac{dy}{dt} = \dfrac{y}{t^2}$, $t \neq 0$. If $\dfrac{dy}{dt} = \dfrac{y}{t^2}$, then $\dfrac{1}{y}\dfrac{dy}{dt} = \dfrac{1}{t^2}$, so

$\displaystyle\int \frac{1}{y}\frac{dy}{dt}\, dt = \int \frac{1}{t^2}\, dt$, i.e., $\displaystyle\int \frac{1}{y}\, dy = \int \frac{1}{t^2}\, dt$, or $\ln|y| = -\dfrac{1}{t} + \bar{c}$.

That is, $|y| = e^{-\frac{1}{t} + \bar{c}}$, or $\boxed{y = c_2 e^{-\frac{1}{t}}}$, $c_2 := \pm e^{\bar{c}}$.

So, a ~~the~~ sol'n to $\begin{cases} \dfrac{dx}{dt} = \dfrac{t}{x^2} \\[2ex] \dfrac{dy}{dt} = \dfrac{y}{t^2} \end{cases}$, $\begin{matrix} x \neq 0 \\ t \neq 0 \end{matrix}$ is $\begin{cases} x = \left(\frac{3}{2}t^2 + c_1\right)^{1/3} \\ y = c_2 e^{-1/t}. \end{cases}$

Example 31.19 $\begin{cases} \dfrac{dx}{dt} = 2e^{2t} \\ \dfrac{dy}{dt} = \dfrac{x^2 - y}{t} \end{cases}, \ t \neq 0.$

Solve $\frac{dx}{dt} = 2e^{2t}$. Int. both sides wrt t:

$$\int \frac{dx}{dt}\, dt = \int 2e^{2t}\, dt$$

$$\boxed{x = e^{2t} + c_1}$$

Rewrite 2^{nd} eq'n: $\dfrac{dy}{dt} = \dfrac{(e^{2t} + c_1)^2 - y}{t} = \dfrac{e^{4t} + 2c_1 e^{2t} + c_1^2 - y}{t}$

$$\frac{dy}{dt} + \frac{1}{t} y = \frac{e^{4t} + 2c_1 e^{2t} + c_1^2}{t}$$

Let $\mu(t) := \exp\left(\int \frac{1}{t}\, dt\right) = \exp\left(\ln|t|\right) = |t|.$

Multiply ODE by t:

$$\underbrace{t \frac{dy}{dt} + y}_{} = e^{4t} + 2c_1 e^{2t} + c_1^2$$

$$\frac{d}{dt}\left[ty\right] = e^{4t} + 2c_1 e^{2t} + c_1^2$$

$$\int \frac{d}{dt}[ty]\, dt = \int e^{4t} + 2c_1 e^{2t} + c_1^2\, dt$$

$$ty = \frac{1}{4} e^{4t} + c_1 e^{2t} + c_1^2 t + c_2$$

$$\boxed{y = \frac{1}{4t} e^{4t} + \frac{c_1}{t} e^{2t} + c_1^2 + \frac{c_2}{t}}$$

So the sol'n to the system was $\begin{cases} x = e^{2t} + c_1 \\ y = \frac{1}{4t} e^{4t} + \frac{c_1}{t} e^{2t} + c_1^2 + \frac{c_2}{t}. \end{cases}$

Lesson 35: turning higher-order ODEs into systems of 1st-order ODEs.

One kind of 2nd-order ODE is

$$y'' = f(y).$$ Indep. var. is t, dep. is y.

$\boxed{\text{Let } x(t) = \frac{dy}{dt}.}$ Then $y'' = \frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{dx}{dt}.$

[Let $x = y'$. Then $x' = y''$.]

So the ODE $y'' = f(y)$ becomes $x' = f(y)$, so

the system equivalent to the eqn is

$$\begin{cases} x = y' \\ x' = f(y) \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{dy}{dt} \\ \frac{dx}{dt} = f(y). \end{cases}$$

Another type of 2nd-order ODE is

$$y'' = f(y, y', t).$$

Let $x(t) = \frac{dy}{dt}$. [i.e., let $x = y'$.] Then $x' = f(y, x, t)$,

So ODE is equivalent to the system:

$$\begin{cases} x = y' \\ x' = f(y, x, t). \end{cases}$$

<u>Example 33.17</u> $y'' = 4y^{-3}$, i.e., $y'' = \frac{4}{y^3}$,

i.e. $y^3 y'' = 4$.

Let $x(t) = y'(t)$, so that $x'(t) = y''(t)$.

Then the ODE becomes $y^3 x'(t) = 4$, so the system of 1^{st}-order ODEs that represents the 2^{nd} order eq'n is

$$\begin{cases} y'(t) = x(t) \\ x'(t) = \frac{4}{y^3} \end{cases}.$$

Multiply 2^{nd} eq'n by $2x$:

$$\underbrace{2x\frac{dx}{dt}}_{=\frac{d}{dt}[x^2]} = \underbrace{\frac{8x}{y^3}}_{=\frac{8}{y^3}\frac{dy}{dt}}$$

so $$\frac{d}{dt}[x^2] = \frac{8}{y^3}\frac{dy}{dt}$$

$$\int \frac{d}{dt}[x^2]\,dt = \int \frac{8}{y^3}\frac{dy}{dt}\,dt$$

$$x^2 = \int \frac{8}{y^3}\,dy = 8\left(-\frac{1}{2}\right)\frac{1}{y^2} + c_1 = \frac{-4}{y^2} + c_1$$

So $\cancel{x = \sqrt{\frac{-2}{y^4} + c_1}}$, and

$y' = x$ ~~becomes~~ $\cancel{y' = \sqrt{\frac{-2}{y^4} + c_1}}$

$$x^2 = \int \frac{8}{y^3}\,dy = 8\left(-\frac{1}{2y^2}\right) + c_1$$

$$x^2 = -\frac{4}{y^2} + c_1$$

$$x = \pm\sqrt{\frac{4}{y^2} + c_1}$$

$$x = \pm\frac{y}{y}\sqrt{\frac{4}{y^2} + c_1}$$

$$= \pm\frac{1}{y}\sqrt{y^2\left(\frac{4}{y^2} + c_1\right)}$$

$\longrightarrow$ $x = \pm\frac{1}{y}\sqrt{4 + c_1 y^2}$, which exists only when $4 + c_1 y^2 \geq 0$, i.e., $y^2 \geq \frac{-4}{c_1}$, so $|y| \geq \sqrt{\frac{-4}{c_1}}$, i.e., $y \geq \sqrt{\frac{-4}{c_1}}$ or $y \leq \sqrt{\frac{4}{c_1}}$

$$y \geq \frac{-2}{\sqrt{c_1}} \quad \text{or} \quad y \leq \frac{2}{\sqrt{c_1}}.$$

The (1st eq'n $y' = x$) becomes $y' = \pm\frac{1}{y}\sqrt{4 + c_1 y^2}$.

$$\int \frac{dy}{dt}\,dt = \int \pm\frac{1}{y}\sqrt{4 + c_1 y^2}\,dt$$

$$\boxed{c_1 y^2 = (c_1 t + c_1 c_2)^2 + 4}$$