Nov. 29: 37A: Review of Power series & Taylor Series.

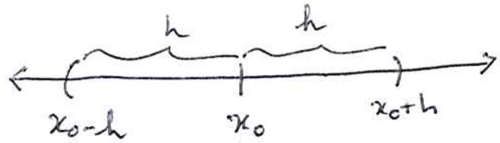
A <u>power series</u> is an infinite series of the form

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

where $a_0, a_1, a_2, \ldots$ are constants (they're called <u>coefficients</u>),
and x_0 is a constant (it's called the <u>center</u> of the power series).

A power series may converge...

1) For only one value of x (i.e., only when $x = x_0$). Int.: $\{x_0\}$
2) For x in a neighborhood of x_0 (i.e., for $x \in \{x : |x-x_0| < h$, and <u>possibly</u> $x = x_0 \pm h\}$). ← Interval of convergence

   (i.e., 

   with endpoints possibly included)
3) For all x, i.e., interval of convergence is $\mathbb{R}$.

To find the interval of convergence, could use...

1) Geometric series test (maybe- depends on a_n)
2) Root test
3) Ratio test

Ex 2 ct'd

i.e., Taylor series is $1 - \frac{1}{2!}(x)^2 + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$.

So, this example tells us that if $\cos(x)$ can be written as a power series centered at $x_0 = 0$, then that power series is $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

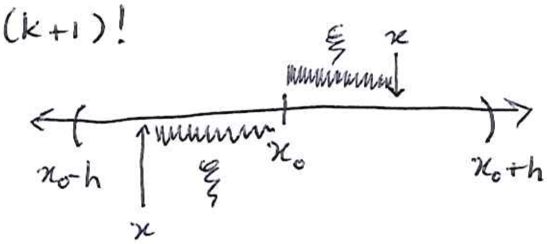
Whether this approximation is actually valid is answered by Taylor's theorem, which involves the <u>remainder</u> of the Taylor series.

<u>Thm. 37.34, p. 536</u>

If a fn. is infinitely diff'ble on an interval $|x - x_0| < h$, then

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} + R_k(x),$$

where $R_k(x) = \dfrac{f^{(k+1)}(\xi)(x - x_0)^{k+1}}{(k+1)!}$, where ξ is between x and x_0.



In addition, if $\lim_{k \to \infty} R_k(x) = 0$, then on $|x - x_0| < h$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Ex ①. $\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n (x-1)^n = 1 + \tfrac{1}{2}(x-1) + \tfrac{1}{4}(x-1)^2 + \tfrac{1}{8}(x-1)^3 + \cdots$

is a geometric series $\displaystyle\sum_{n=0}^{\infty} a \cdot r^n$, with $a = 1$ and $r = \frac{1}{2}(x-1)$,

and so it converges when $\underline{|r| < 1}$, i.e.,
(no matter what a is)

$\left|\frac{1}{2}(x-1)\right| < 1$, i.e., $|x-1| < 2$, i.e., $-2 < x-1 < 2$, i.e.,

$-1 < x < 3$. Moreover, the sum of the "geometric" power series is $\dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{2}(x-1)} = \dfrac{2}{2-(x-1)} = \dfrac{2}{3-x}$.

So, $\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n (x-1)^n = \frac{2}{3-x}$, for $x \in (-1, 3)$.

We still need to test at the endpoints $\{-1, 3\}$.

$\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n (-1-1)^n = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n (-2)^n = \sum_{n=0}^{\infty} (-1)^n$, which

has the partial sums

$$\begin{aligned} S_0 &= (-1)^0 = 1 \\ S_1 &= (-1)^0 + (-1)^1 = 0 \\ S_2 &= S_1 + (-1)^2 = 1 \\ &\vdots \end{aligned}$$

, i.e.,

the sequence of partial sums is $\{1, 0, 1, 0, 1, 0, \ldots\}$, which diverges — so $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ diverges too.

Ex ①, ct'd.

Check $x = 3$: $\sum_{n=0}^{\infty} (\frac{1}{2})^n (3-1)^n = \sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + \cdots$,

which has partial sums $\{1, 2, 3, 4, 5, \ldots\}$. The seq. of partial sums diverges, and so $\sum_{n=0}^{\infty} 1^n$ diverges too.

So the interval of convergence remains open at both endpoints, and our concluding statement becomes

$$\sum_{n=0}^{\infty} (\tfrac{1}{2})^n (x-1)^n = \frac{2}{3-x} \quad \text{only when } x \in (-1, 3),$$

and diverges elsewhere.

---

p. 534, Thm. 37.24: For a function $f(x)$ that can be defined as the sum of a power series for some x (situations 1, 2, and 3),

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(x_0)}{n!}}_{a_n} (x - x_0)^n.$$

} "Taylor series generated by $f(x)$ around / "centered at" $x = x_0$"

Ex ② Find the Taylor series generated by $\cos(x)$ around $x_0 = 0$.

$$f^{(0)}(x) = \cos(x) \Rightarrow f^{(0)}(0) = \cos(0) = 1$$
$$f^{(1)}(x) = -\sin(x) \Rightarrow f^{(1)}(0) = -\sin(0) = 0$$
$$f^{(2)}(x) = -\cos(x) \Rightarrow f^{(2)}(0) = -\cos(0) = -1$$
$$f^{(3)}(x) = \sin(x) \Rightarrow f^{(3)}(0) = \sin(0) = 0$$
$$f^{(4)}(x) = \cos(x) \Rightarrow f^{(4)}(0) = \cos(0) = 1.$$

Taylor series is $\underbrace{\frac{1}{0!}}_{0! = 1} + \frac{0}{1!}(x-0) - \frac{1}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4 + \frac{0}{5!}(x-0)^5 - \frac{1}{6!}(x-0)^6 + \cdots$