1 DEC. : 37B: (Taylor) Series Sol'n Methods.

- A fn. whose Taylor remainders R_k are s.t. $\lim_{k\to\infty} R_k = 0$ is called an <u>analytic</u> function.
- p. 537 – formal def'n of analyticity <u>at a point</u>, <u>on an interval</u>, etc.

<u>Examples</u>

$$\left.\begin{aligned} &\cdot\ e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &\cdot\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &\cdot\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}\right\}, \text{ for } x \in \mathbb{R}.$$

- $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, for $x \in [-1, 1]$

Recall

~~Ea~~ <u>any</u> <u>convergent</u> power series is diff'ble & integrable <u>termwise</u>, i.e., if $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, then

$$f'(x) = \frac{d}{dx}\left[\sum_{m=0}^{\infty} a_m (x-x_0)^m\right] \underset{\text{Convergence}}{=} \sum_{m=0}^{\infty} \frac{d}{dx}\left[a_m (x-x_0)^m\right]$$

$$= \sum_{m=0}^{\infty} m a_m (x-x_0)^{m-1},$$

and

$$\int f(x)\,dx = \int \left[\sum_{m=0}^{\infty} a_m (x-x_0)^m\right] dx \underset{\text{Convergence}}{=} \sum_{m=0}^{\infty} \left[\int a_m (x-x_0)^m\,dx\right]$$

$$= \sum_{m=0}^{\infty} \frac{a_m}{m+1} (x-x_0)^{m+1}.$$

General form of a linear n^{th} order (non-homogeneous) ODE:

$$y^{(n)} + f_{n-1}(x)\, y^{(n-1)} + f_{n-2}(x)\, y^{(n-2)} + \cdots + f_1(x)\, y' + f_0(x)\, y = Q(x)$$

We had Thm. 19.2 to guarantee existence & uniqueness of sol'ns when $f_i(x)$ are all cts. on a common interval of $\mathbb{R}$.

Thm. 37.51 guarantees existence & uniqueness and analyticity of the sol'n when $f_i(x)$ are all analytic at some point $x = x_0$.

Even if we can't find the sol'n, we're guaranteed that it exists (& is unique), and we can seek its Taylor series representation.

Ex 37.541 Find a particular sol'n of

$$y'' - (x+1)\, y' + x^2 y = x$$

$$y'' + (-x-1)\, y' + x^2 y = x$$

subject to the initial cond'ns $y(0) = 1$, $y'(0) = 1$.

Gen. form: $y'' + f_1(x)\, y' + f_0(x)\, y = Q(x)$

In this example, $f_1(x) = -x-1$, $f_0(x) = x^2$, $Q(x) = x$.

Recall: Polynomials are their own Taylor series, so they're analytic.

So, in this example, we're guaranteed a unique analytic sol'n.

Ex., ct'd.

Write $y(x) = \underbrace{y(0)}_{1} + \underbrace{y'(0)}_{1}x + \frac{\overbrace{y''(0)}^{?\ 1}}{2!}x^2 + \frac{\overbrace{y'''(0)}^{?}}{3!}x^3 + \cdots$

$= \sum_{m=0}^{\infty} \frac{y^{(m)}(0)}{m!} x^m$, which we know exists and equals y, by Thm. 37.51.

The initial cond'ns will give us values of the coefficients.

That is, $y(0) = 1$

$y'(0) = 1$.

To find $y''(0)$, use the ODE:

$$y'' - (x+1)y' + x^2 y = x$$

$$y''(0) - (0+1)\underbrace{y'(0)}_{1} + (0)^2\underbrace{y(0)}_{1} = 0$$

$$y''(0) - 1 = 0 \ , \quad \text{so} \quad y''(0) = 1.$$

To find $y'''(0)$, use the ODE:

$$\frac{d}{dx}\left[y'' - (x+1)y' + x^2 y\right] = \frac{d}{dx}[x]$$

$$y''' - \frac{d}{dx}(x+1)y' - (x+1)y'' + \frac{d}{dx}(x^2)y + x^2 y' = 1$$

$$y''' - y' - (x+1)y'' + 2xy + x^2 y' = 1$$

Subst. $x=0$: $\quad y'''(0) - \underbrace{y'(0)}_{=1} - (\overset{0}{x}+1)\underbrace{y''(0)}_{=1} + 2(0)\underbrace{y(0)}_{=1} + 0^2\underbrace{y'(0)}_{=1} = 1$

$$y'''(0) - 1 - 1 = 1$$

$$y'''(0) = 3.$$

To find $y^{IV}(0)$, differentiate the ODE:

$$\frac{d}{dx}\left[y''' - (x+1)y'' + (x^2-1)y' + 2xy \right] = \frac{d}{dx}[1]$$

$$y^{IV} - \frac{d}{dx}(x+1)y'' - (x+1)y''' + \frac{d}{dx}(x^2-1)y' + (x^2-1)y'' + \frac{d}{dx}(2x)\,y + 2xy' = 0$$

$$y^{IV} - y'' - (x+1)y''' + 2xy' + (x^2-1)y'' + 2y + 2xy' = 0$$

Subst.
$x=0$:

$$y^{IV}(0) - \underbrace{y''(0)}_{1} - (0+1)\underbrace{y'''(0)}_{3} + 4(0)\underbrace{y'(0)}_{1} + (0^2-1)\underbrace{y''(0)}_{1} + 2\underbrace{y(0)}_{1} = 0$$

$$y^{IV}(0) - 1 - 3 - 1 + 2 = 0$$

$$y^{IV}(0) = 1 + 3 + 1 - 2 = 3.$$

Substituting all derivs of y evaluated at 0 into the Taylor representation of y:

$$y(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{IV}(0)}{4!}x^4 + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{\cancel{4}\,8} + \cdots$$

$$\frac{3}{4!} = \frac{3}{4\cdot\cancel{3}\cdot 2}$$

Another method using series:

Example 37.6 $\quad y'' - (x+1)y' + x^2 y = x, \quad y(0) = 1, \quad y'(0) = 1.$

By Thm. 37.51, we're guaranteed a unique analytic sol'n:

$$y(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

To substitute $y(x)$, written as its Taylor series, into the ODE, we need t/compute y' & y''.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\Rightarrow \quad y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$$

$$\Rightarrow \quad y''(x) = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + 5\cdot 4\, a_5 x^3 + \dots$$

$$\left[2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots\right] - (x+1)\left[\overset{-(x+1)}{a_1} + \overset{-(x+1)}{2a_2 x} + \overset{-(x+1)}{3a_3 x^2} + \overset{-(x+1)}{4a_4 x^3} + \dots\right] +$$

$$+ x^2\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right] = x$$

$$(2a_2 - a_1) + x\,(6a_3 - a_1 - 2a_2) + x^2\,(12a_4 - 2a_2 - 3a_3 + a_0) +$$

$$+ x^3\,(20a_5 - 3a_3 - 4a_4 + a_1) + \dots = x$$

Because $\{1, x, x^2, x^3, \dots\}$ is a lin. indep. set, the only sol'n here is to equate coeffs. of like powers of x:

$$2a_2 - a_1 = 0$$

$$6a_3 - a_1 - 2a_2 = 1$$

$$12a_4 - 2a_2 - 3a_3 + a_0 = 0, \qquad 20a_5 - 3a_3 - 4a_4 + a_1 = 0, \dots$$

This gives us a system of linear (algebraic) eq'ns to solve for $a_0, a_1, a_2, a_3, \ldots$

Solve this in any way you like ...

Obtain $a_0 = 1$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{2}$, $a_4 = \frac{1}{8}$, ...

Get $y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{8}x^4 + \cdots$.