Euler Eq'ns. DEC 06

Form: $a_n (x-x_0)^n y^{(n)} + \cdots + a_2 (x-x_0)^2 y'' + a_1 (x-x_0) y' + a_0 y = Q(x)$

where $Q(x)$ is cts., and a_i & x_0 are consts.

Example: $(x-3)^2 y'' + (x-3) y' + y = 0$

Substitution (general): let $x - x_0 := e^u$, or

$$u := \ln(x - x_0)$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{d}{dx}\left[\ln(x-x_0)\right]$$

$$= \frac{dy}{du} \cdot \left(\frac{1}{x-x_0}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{du}\left(\frac{1}{x-x_0}\right)\right] \underset{\text{chain}}{=} \frac{d}{du}\left[\frac{dy}{du}\left(\frac{1}{x-x_0}\right)\right]\frac{du}{dx} =$$

$$= \frac{d^2y}{du^2}\left(\frac{1}{x-x_0}\right)\frac{du}{dx} = \frac{d^2y}{du^2}\left(\frac{1}{x-x_0}\right)^2$$

$$\frac{d^3y}{dx^3} = \frac{d^3y}{du^3}\left(\frac{1}{x-x_0}\right)^3 \quad \ldots \qquad \frac{d^n y}{(dx)^n} = \frac{d^n y}{du^n}\left(\frac{1}{x-x_0}\right)^n$$

$$a_m (x-x_0)^n \frac{d^n y}{dx^n} + a_{n-1}(x-x_0)^{n-1}\frac{d^{n-1}y}{dx^n} + \cdots + a_1 (x-x_0)\frac{dy}{dx} + a_0 y = Q(x)$$

$$a_m \cancel{(x-x_0)^n} \frac{d^n y}{du^n}\cancel{\left(\frac{1}{x-x_0}\right)^n} + \cdots + a_1 \cancel{(x-x_0)}\frac{dy}{du}\cancel{\left(\frac{1}{x-x_0}\right)} + a_0 y = Q(e^u + x_0)$$

$$a_m \frac{d^n y}{du^n} + \cdots + a_2 \frac{d^2 y}{du^2} + a_1 \frac{dy}{du} + a_0 y = Q(e^u + x_0)$$

indep. variable is u, dep. var. is y.

<u>Example</u> : Solve :

$$(x-3)^2 y'' + (x-3) y' + y = 0 \quad , \quad \text{for } x \neq 3 . \quad (x > 3 ?)$$

let $e^u := x-3$, so $u = \ln(x-3)$.

eq'n becomes

$$\cancel{(x-3)^2}\frac{d^2y}{du^2}\cancel{\left(\frac{1}{x-3}\right)^2} + \cancel{(x-3)}\frac{dy}{du}\cancel{\left(\frac{1}{x-3}\right)} + y = 0$$

$$\frac{d^2y}{du^2} + \frac{dy}{du} + y = 0. \qquad [\, y'' + y' + y = 0 \,]$$

Let $y(u) = e^{p \cdot u}$, some p. Substitute into eq'n, solve for p:

$$e^{p \cdot u}\underbrace{\left(p^2 + p + 1\right)}_{\text{characteristic eq'n.}} = 0$$

The discriminant of the char. eq'n is

$$1^2 - 4(1)(1) = 1-4 = -3,$$

which is negative, so we have two complex sol'ns.

$$p = \frac{-1 \pm \sqrt{-3}}{2\cdot 1} = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}$$

$$= -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

sol'ns are the 2-param. family:

$$y(u) = e^{-\frac{1}{2}u}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}u\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}u\right)\right)$$

Back-substitute: $u = \ln(x-3)$:

$$y(x) = \underbrace{e^{-\frac{1}{2}\ln(x-3)}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln(x-3)\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln(x-3)\right)\right]$$

$$\longmapsto e^{-\frac{1}{2}\ln(x-3)} = \exp\left(\ln\left((x-3)^{-1/2}\right)\right) = (x-3)^{-1/2}$$

$$\longmapsto a\cdot\ln(b) = \ln(b^a)$$

So $$y(x) = \frac{1}{\sqrt{x-3}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln(x-3)\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln(x-3)\right)\right].$$

Laplace Transforms.

<u>GOAL</u>: Transform an initial value problem into an algebraic eq'n using the Laplace transform of the unknown fn.

<u>Def.</u> For $f : \mathbb{R}^+ \to \mathbb{R}$, defined for $t > 0$, let $s \in \mathbb{R}$, and define

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt,$$

only for values of s where the integral exists/converges.

$F(s)$ is called the <u>Laplace transform</u> of $f(t)$, and is written as $F(s) = \mathcal{L}\{f(t)\}$ or $F = \mathcal{L}\{f\}$.

Ex. $f(t) = 1$, $t > 0$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = F(s) := \int_0^\infty e^{-st} \cdot 1 \, dt .$$

$$F(s) = \int_0^\infty e^{-st}\, dt = \lim_{b\to\infty} \int_0^b e^{-st}\, dt$$

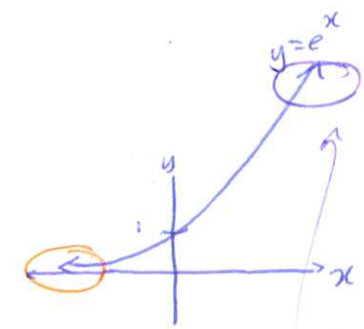
$$\frac{d}{dt}\left[-\frac{1}{s} e^{-st}\right] = \cancel{-\frac{1}{s}}(\cancel{-s}) e^{-st} = e^{-st}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{s} e^{-st} \Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{1}{s} e^{-sb} + \frac{1}{s} e^{-s\cdot 0} \right]$$

$$= \lim_{b\to\infty} \left[-\frac{1}{s} e^{-sb} + \frac{1}{s} \right] .$$

If $s > 0$, then $\lim_{b\to\infty} e^{-sb} = 0$.



If $s = 0$, then $\lim_{b\to\infty} e^{-sb} = 1$

If $s < 0$, then $\lim_{b\to\infty} e^{-sb} = +\infty$ (doesn't exist)

So $F(s) = \begin{cases} \frac{1}{s}, & s > 0 \\ \cancel{0, \quad s = 0} \\ +\infty \end{cases}$ doesn't make sense in the original defn.

$F(s) = \frac{1}{s}$, or $\mathcal{L}\{1\} = \frac{1}{s}$, $s > 0$.

Example. $f(t) = t \quad , \quad t > 0.$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t \, dt = \lim_{b\to\infty} \int_0^b e^{-st}\, \underline{t} \, dt$$

$\int u \, dv = uv - \int v \, du$

let $u := t$

$du = dt$

$dv := e^{-st} dt$

$v = -\frac{1}{s} e^{-st}$

$$= \lim_{b\to\infty} \left[-\frac{t}{s} e^{-st} \Big|_0^b - \int_0^b -\frac{1}{s} e^{-st} \, dt \right]$$

$$= \lim_{b\to\infty} \left[-\frac{t}{s} e^{-st} \Big|_0^b + \frac{1}{s} \int_0^b e^{-st} \, dt \right]$$

$$= \lim_{b\to\infty} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[\left(-\frac{1}{s} e^{-st}\right)\left(t + \frac{1}{s}\right) \Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[\left(-\frac{1}{s} e^{-s\cdot b} \left(b + \frac{1}{s}\right)\right) - \left(-\frac{1}{s} e^{-s\cdot 0}\right)\left(0 + \frac{1}{s}\right) \right]$$

Notice: if $s < 0$, then $e^{-sb} \xrightarrow[b\to\infty]{} \infty$ and so entire expression $-\frac{1}{s} e^{-sb}\left(b + \frac{1}{s}\right) \to \infty$.

if $s = 0$, then $\int_0^\infty e^{-st}\, t \, dt = \int_0^\infty t \, dt \to \infty$

if $s > 0$, then:

$$\lim_{b\to\infty} -\frac{1}{s} e^{-sb}\left(b + \frac{1}{s}\right) = \lim_{b\to\infty} \frac{-\left(b + \frac{1}{s}\right)}{e^{sb}} \to \text{“}\frac{\infty}{\infty}\text{”}$$

indet. form, so use l'Hôp. rule

$$= \lim_{b\to\infty} \frac{-1}{s e^{sb}} = 0.$$

So $\mathcal{L}\{t\} = 0 - \left(-\frac{1}{s}e^{-s\cdot 0}\right)\left(0+\frac{1}{s}\right) \quad , \quad s>0$

$= \frac{1}{s^2} \quad , \quad s>0.$