Laplace Transforms, ct'd.

For $f(t): \mathbb{R}^+ \to \mathbb{R}$, $\mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)\, dt$, for all values of s where the improper integral converges.

The L.T. of a fn. of t is itself a function of s.

$\mathcal{L}\{1\}$, $\mathcal{L}\{t\}$.

$= 1/s$, $s>0$ $\qquad = \frac{1}{s^2}$, $s>0$.

$e^{a+b} = e^a e^b$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{t(a-s)}\, dt = \lim_{b\to\infty} \int_0^b e^{t(a-s)}\, dt =$$

$$= \lim_{b\to\infty} \left[\frac{1}{(a-s)} e^{t(a-s)} \Big|_0^b \right] = \lim_{b\to\infty} \left[\frac{1}{a-s} \left(e^{b(a-s)} - e^{0(a-s)} \right) \right] =$$

$$\stackrel{?}{=} \frac{1}{a-s} \underbrace{\lim_{b\to\infty} \left(e^{b(a-s)} \right)}_{=0,\text{ if } a-s<0,\text{ i.e., } s>a.} - \frac{1}{a-s}$$

Case I. $a-s>0$: $\lim_{b\to\infty} e^{b(a-s)} = +\infty$ (does not exist)

Case II. $a-s<0$: $\lim_{b\to\infty} e^{b(a-s)} = 0$

Case III $a-s=0$: $\lim_{b\to\infty} e^{b(a-s)} = 1$, but $\int_0^\infty e^{t(a-s)}\, dt = \int_0^\infty 1\cdot dt$, which diverges.

$$\mathcal{L}\{e^{at}\} = \frac{1}{a-s}(0) - \frac{1}{a-s} = \frac{1}{s-a}, \text{ if } s>a.$$

ef. The <u>Gamma Function</u> is defined as:

$$\Gamma(x) := \int_0^\infty e^{-t}\, t^{x-1}\, dt, \quad \text{for } x > 0.$$

A (simpler?) recursion formula: $\Gamma(1) = 1$

$\Gamma(x+1) = x\,\Gamma(x)$.

$$\Gamma(m) = (m-1)\Gamma(m-1) = (m-1)(m-2)\Gamma(m-2) = \cdots = (m-1)!$$

So if $m \in \mathbb{N}$, $\Gamma(m) = \cancel{m!}\ (m-1)!$.

Example.

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st}\, t^a\, dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \left(\frac{1}{s}\right) du = \left(\frac{1}{s}\right)^{a+1} \int_0^\infty e^{-u} u^a\, du$$

let $u := st$, so $t = \frac{u}{s}$ and $dt = \frac{1}{s}\, du$.

Specify: $s > 0$. Then $u(0) = s \cdot 0 = 0$, $u(\infty) = s \cdot \infty = \infty$

Observe: $\Gamma(a+1) = \int_0^\infty e^{-t}\, t^{a+1-1}\, dt = \int_0^\infty e^{-t} t^a\, dt = \int_0^\infty e^{-u} u^a\, du$

So $\mathcal{L}\{t^a\} = \left(\frac{1}{s}\right)^{a+1} \Gamma(a+1)$, $a > 0$, $s > 0$.

$\mathcal{L}\{t^a\} = \left(\frac{1}{s}\right)^{a+1} (a)!$, if $a \in \mathbb{N}$

<u>For example</u>: $\mathcal{L}\{t\} = \mathcal{L}\{t^1\} \overset{?}{=} \frac{1}{s^2}$.

$\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$.

$\mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

$\vdots$

(1) L.T. is a linear transformation:

$$\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

This prop'y follows from the defn of a L.T., and from the linearity of the convergent improper integral.

Example:

$$\cosh(kt) = \frac{1}{2}\left(e^{kt} + e^{-kt}\right)$$
$$\sinh(kt) = \frac{1}{2}\left(e^{kt} - e^{-kt}\right)$$

$$\mathcal{L}\{\cosh(kt)\} = \mathcal{L}\left\{\frac{1}{2}\left(e^{kt} + e^{-kt}\right)\right\}$$

LINEARITY.

$$= \frac{1}{2}\left(\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right)$$
$$= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right)$$
$$= \frac{1}{2}\left(\frac{(s+k) + (s-k)}{(s-k)(s+k)}\right)$$
$$= \frac{1}{2}\left(\frac{2s}{s^2 - k^2}\right) = \frac{s}{s^2 - k^2}.$$

② Existence.

If $f(t)$ is piecewise cts. for $t \geq 0$,

and if $f(t)$ is of exponential order – i.e.,

if $\exists M, c, T > 0$, s.t. $\forall t \geq T$, $|f(t)| \leq Me^{ct}$ ...

... Then $\mathcal{L}\{f(t)\}$ exists for all $s > c$.

③ Inverses & uniqueness of the inverse.

Remember: $\mathcal{L}\{f(t)\} = F(s)$ was notation we used in the beginning.

$\mathcal{L}^{-1}\{F(s)\} = f(t)$.

If $F(s) = G(s)$ $\forall s > c$, some $c \in \mathbb{R}$,

Then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{G(s)\}$ whenever on $[0, +\infty)$,

each of $\mathcal{L}^{-1}\{F(s)\}$ and $\mathcal{L}^{-1}\{G(s)\}$ are both cts.

So, two fns. tht. are p/w cts. & of exponential order with the same L.T. can differ only at their (isolated) pts. of discontinuity.

For practical purposes, $\mathcal{L}^{-1}\{F(s)\}$ is unique.

IDEA: Transform both sides of an ODE
Solve for $\mathcal{L}\{y\}$
Inverse transform: $\mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = y$.

$\int u\,dv = uv - \int v\,du$

$u = e^{-st}$ $\quad v = f(t)$

$du = -s e^{-st}\,dt$ $\quad dv = f'(t)\,dt$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{b\to\infty} \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{b\to\infty} \left[e^{-st} f(t) \Big|_{t=0}^{b} - \int_0^b -s e^{-st} f(t)\,dt \right]$$

$$= \lim_{b\to\infty} \left[e^{-sb} f(b) - f(0) + s \int_0^b e^{-st} f(t)\,dt \right]$$

if $s > 0$ ...

$$= -f(0) + s \lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt$$

$$= s\mathcal{L}\{f(t)\} - f(0), \quad s > 0$$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left[s\mathcal{L}\{f(t)\} - f(0) \right] - f'(0)$$

$$= s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0), \quad s > 0.$$

$\vdots$

Example. $y'' - y' - 6y = 0$, $y(0) = 2$, $y'(0) = -1$

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - \left[s\mathcal{L}\{y\} - y(0)\right] - 6\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - 2s + 1 - s\mathcal{L}\{y\} + 2 - 6\mathcal{L}\{y\} = 0$$

$$\mathcal{L}\{y\}(s^2 - s - 6) = 2s - 3$$

$$\mathcal{L}\{y\} = \frac{2s-3}{s^2-s-6}, \quad \text{so} \quad y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\}$$

Partial Fractions: $\dfrac{2s-3}{s^2-s-6} = \dfrac{2s-3}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2}$

$$= \frac{3/5}{s-3} + \frac{7/5}{s+2}$$

$$y = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{3/5}{s-3} + \frac{7/5}{s+2}\right\}$$

$$= \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\boxed{y(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}.}$$