DEMO TOPIC 4: EXACT ODE.

Example 1 $\quad (y-4x)\,dx + (x-6y)\,dy = 0$

Check: $\dfrac{\partial}{\partial y}(y-4x) \stackrel{?}{=} \dfrac{\partial}{\partial x}(x-6y)$

$$1 \stackrel{?}{=} 1 \quad \checkmark$$

Yes, the eq'n is exact.

$$F(x,y) := \int y - 4x \; dx$$

$$= xy - 2x^2 + g(y).$$

$$\frac{\partial F}{\partial y} := x - 6y$$

$$\frac{\partial}{\partial y}\left[xy - 2x^2 + g(y)\right] := x - 6y$$

$$x + g'(y) = x - 6y$$

$$g'(y) = -6y$$

$$\int g'(y)\,dy = \int -6y\,dy$$

$$g(y) = -3y^2 + C$$

So $F(x,y) = xy - 2x^2 - 3y^2 + C$, and $F(x,y) = 0$ solves the ODE.

Example ①, ctd.

Check: $xy - 2x^2 - 3y^2 + c = 0$ solves the ODE

$$(y-4x)\,dx + (x-6y)\,dy = 0 \iff y - 4x + (x-6y)\frac{dy}{dx} = 0$$

Implicit:

$$\frac{d}{dx}\left[xy - 2x^2 - 3y^2 + c\right] = \frac{d}{dx}[0]$$

$$\frac{d}{dx}[x]\,y + x\frac{d}{dx}[y] - \frac{d}{dx}[2x^2] - \frac{d}{dy}[3y^2]\frac{d}{dx}[y] = 0$$

$$y + x\frac{dy}{dx} - 4x - 6y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 6y\frac{dy}{dx} = 4x - y$$

$$\frac{dy}{dx}(x-6y) = 4x - y$$

$$\frac{dy}{dx} = \frac{4x-y}{x-6y}$$

Check: $y - 4x + (x-6y)\dfrac{dy}{dx} =$

$$= y - 4x + \cancel{(x-6y)}\left(\frac{4x-y}{\cancel{x-6y}}\right)$$

$$= \cancel{y} - \cancel{4x} + 4x - y = 0 \ \checkmark$$

DEMO: Integrating factor.

Example (2) $y' + ay = b$.

- Let $\mu(x) := e^{\int a \, dx} = e^{ax}$

- Multiply ODE by $\mu(x)$:

$$\mu(x) y' + \mu(x) a y = \mu(x) b$$

$$\boxed{e^{ax} y' + e^{ax} a y} = e^{ax} b$$

- Notice the LHS is the derivative of

$y \cdot \mu(x)$ w.r.t. x:

$$=$$

$$\frac{d}{dx}\left[y \cdot \mu(x)\right] = \frac{d}{dx}\left[y e^{ax}\right] = \frac{d}{dx}[y]\, e^{ax} + y \frac{d}{dx}[e^{ax}]$$

$$= \boxed{e^{ax} \frac{dy}{dx} + a e^{ax} y}$$

$$\frac{d}{dx}\left[e^{ax} y\right] = e^{ax} b$$

- Integrate w.r.t. x:

$$\int \frac{d}{dx}\left[e^{ax} y\right] dx = \int e^{ax} b \, dx$$

$$e^{ax} y = \frac{b}{a} e^{ax} + C$$

Example ② ct'd.

$$e^{ax} y = \frac{b}{a} e^{ax} + C$$

$$\cancel{e^{-ax} e^{ax}} y = \frac{b}{a} \cancel{e^{ax} e^{-ax}} + Ce^{-ax}$$

$$y = \frac{b}{a} + Ce^{-ax}.$$

- Check that $y = \frac{b}{a} + Ce^{-ax}$ solves $y' + ay = b$.

$$y' = \frac{d}{dx}\left[\frac{b}{a} + Ce^{-ax}\right]$$

$$= -aCe^{-ax}$$

So $$y' + ay = \left(-aCe^{-ax}\right) + a\left(\frac{b}{a} + Ce^{-ax}\right)$$

$$= \cancel{-aCe^{-ax}} + b + \cancel{aCe^{-ax}}$$

$$= b. \checkmark$$

Example (3), Prob. 10 from Exercise II on p. 97

$$L \frac{di}{dt} + Ri = E \sin(kt).$$

Want the form: $\dfrac{di}{dt} + f(t)\, i = g(t)$.

- So, divide by L:

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \sin(kt)$$

- Let $\mu(t) := e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$.

- Multiply ODE by $\mu(t)$:

$$e^{\frac{R}{L}t} \frac{di}{dt} + \frac{R}{L} e^{\frac{R}{L}t} i = \frac{E}{L} \sin(kt)\, e^{\frac{R}{L}t}$$

- Observe the LHS $= \dfrac{d}{dt}\left[i e^{\frac{R}{L}t} \right] = e^{\frac{R}{L}t} \dfrac{di}{dt} + \dfrac{R}{L} e^{\frac{R}{L}t} i$

- Antidifferentiate both sides w.r.t. t

$$\int \frac{d}{dt}\left[i e^{\frac{R}{L}t} \right] dt = \int \frac{E}{L} \sin(kt)\, e^{\frac{R}{L}t}\, dt$$

$$i e^{\frac{R}{L}t} = \int \frac{E}{L} \sin(kt)\, e^{\frac{R}{L}t}\, dt$$

Aside: $\int \frac{E}{L}\sin(kt)e^{\frac{R}{L}t}\,dt$. To integrate by parts, use $\int u\,dv = uv - \int v\,du$

Choose $u := \frac{E}{L}\sin(kt)$ | $v = \frac{L}{R}e^{\frac{R}{L}t}$

$du = \frac{kE}{L}\cos(kt)\,dt$ | $dv := e^{\frac{R}{L}t}\,dt$.

$$\int \frac{E}{L}\sin(kt)e^{\frac{R}{L}t}\,dt = \frac{E}{L}\cdot\frac{L}{R}\sin(kt)e^{\frac{R}{L}t} - \int \frac{L}{R}\frac{kE}{L}\cos(kt)\cdot e^{\frac{R}{L}t}\,dt$$

$$\int \frac{E}{L}\sin(kt)e^{\frac{R}{L}t}\,dt = \frac{E}{R}\sin(kt)e^{\frac{R}{L}t} - \underline{\int \frac{kE}{R}\cos(kt)e^{\frac{R}{L}t}\,dt}.$$

Let $u := \frac{kE}{R}\cos(kt)$ $\quad v = \frac{L}{R}e^{\frac{R}{L}t}$

$du = -\frac{k^2E}{R}\sin(kt)$. $\quad dv := e^{\frac{R}{L}t}\,dt$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{E}{L}\sin(kt)e^{\frac{R}{L}t}\,dt = \frac{E}{R}\sin(kt)e^{\frac{R}{L}t} - \left(\frac{kE}{R}\cdot\frac{L}{R}\cos(kt)e^{\frac{R}{L}t} - \int -\frac{k^2E}{R}\frac{L}{R}\sin(kt)e^{\frac{R}{L}t}\,dt\right)$$

$$\frac{E}{L}\int \sin(kt)e^{\frac{R}{L}t}\,dt = \frac{E}{R}\sin(kt)e^{\frac{R}{L}t} - \frac{kEL}{R^2}\cos(kt)e^{\frac{R}{L}t} - \frac{k^2EL}{R^2}\int \sin(kt)e^{\frac{R}{L}t}\,dt$$

$$\frac{E}{L}\int \sin(kt)e^{\frac{R}{L}t}\,dt + \frac{k^2EL}{R^2}\int \sin(kt)e^{\frac{R}{L}t}\,dt = \frac{E}{R}\sin(kt)e^{\frac{R}{L}t} - \frac{kEL}{R^2}\cos(kt)e^{\frac{R}{L}t}$$

So $$\left(\frac{E}{L} + \frac{k^2 EL}{R^2}\right)\int \sin(kt)\, e^{\frac{R}{L}t}\, dt = \frac{E}{R}\sin(kt)\, e^{\frac{R}{L}t} - \frac{kEL}{R^2}\cos(kt)\, e^{\frac{R}{L}t}$$

$$\int \sin(kt)\, e^{\frac{R}{L}t}\, dt = \frac{\frac{E}{R}\sin(kt)\, e^{\frac{R}{L}t} - \frac{kEL}{R^2}\cos(kt)\, e^{\frac{R}{L}t}}{\frac{E}{L} + \frac{k^2 EL}{R^2}}$$

Aside to the aside: $$\frac{E}{L} + \frac{k^2EL}{R^2} = \frac{ER^2 + k^2EL^2}{LR^2} = \frac{E(R^2 + k^2L^2)}{LR^2}$$

So $$\frac{1}{\frac{E}{L} + \frac{k^2EL}{R^2}} = \frac{LR^2}{E(R^2 + k^2L^2)}.$$

$$\int \sin(kt)\, e^{\frac{R}{L}t}\, dt = \frac{LR^{\cancel{2}}}{\cancel{E}(R^2 + k^2L^2)}\left(\frac{\cancel{E}}{\cancel{R}}\, e^{\frac{R}{L}t}\right)\left(\sin(kt) - \frac{kL}{R}\cos(kt)\right)$$

$$= \boxed{\frac{LR}{k^2L^2 + R^2}\, e^{\frac{R}{L}t}\left(\sin(kt) - \frac{kL}{R}\cos(kt)\right)}.$$

From our solution, $$ie^{\frac{R}{L}t} = \int \frac{E}{L}\sin(kt)\, e^{\frac{R}{L}t}\, dt$$

$$= \frac{E}{L}\boxed{\int \sin(kt)\, e^{\frac{R}{L}t}\, dt}.$$

So $$ie^{\frac{R}{L}t} = \frac{E}{\cancel{L}}\cdot\frac{\cancel{L}R}{(k^2L^2 + R^2)}\, e^{\frac{R}{L}t}\left(\sin(kt) - \frac{kL}{R}\cos(kt)\right) + C$$

$$i(t) = \frac{ER}{k^2L^2 + R^2}\left(\sin(kt) - \frac{kL}{R}\cos(kt)\right) + Ce^{-\frac{R}{L}t}.$$

Ex ③, cont'd.

Check: $i(t) = \boxed{\frac{ER}{k^2L^2+R^2}\left(\sin(kt) - \frac{kL}{R}\cos(kt)\right) + ce^{-\frac{R}{L}t}}$

solves $L\boxed{\frac{di}{dt}} + R\boxed{i} = E\sin(kt)$.

$$\frac{di}{dt} = \boxed{\frac{ER}{k^2L^2+R^2}\left(k\cos(kt) + \frac{k^2L}{R}\sin(kt)\right) - \frac{cR}{L}e^{-\frac{R}{L}t}}$$

$$L\frac{di}{dt} + Ri = \frac{ERL}{k^2L^2+R^2}\left(k\cos(kt) + \frac{k^2L}{R}\sin(kt)\right) \cancel{- cRe^{-\frac{R}{L}t}} +$$

$$+ \frac{ER^2}{k^2L^2+R^2}\left(\sin(kt) - \frac{kL}{R}\cos(kt)\right) \cancel{+ cRe^{-\frac{R}{L}t}}$$

$$= \sin(kt)\left(\frac{ER^2}{k^2L^2+R^2} + \frac{k^2L}{\cancel{R}}\left(\frac{E\cancel{R}L}{k^2L^2+R^2}\right)\right) + \cos(kt)\left(\frac{kERL}{k^2L^2+R^2} - \frac{kERL}{\cancel{R}(k^2L^2+R^2)}\right)$$

$$= \sin(kt)\left(\frac{ER^2 + k^2EL^2}{k^2L^2+R^2}\right) + \cancel{\cos(kt)(0)} \to 0$$

$$= \sin(kt)\cdot E\left(\cancel{\frac{R^2+k^2L^2}{R^2+k^2L^2}}\right) \to 1 \quad \stackrel{?}{=} E\sin(kt)$$

$$= \sin(kt)\cdot E\left(\frac{R^2+k^2L^2}{R^2+k^2L^2}\right)$$

$= E\sin(kt)$, which satisfies the ODE.

DEMO PRACTICE: SUBST. / MULT.

Example (4), Prob. 1 from p. 103

$$2xy\,\frac{dy}{dx} + (1+x)\,y^2 = e^x.$$

$$\left.\begin{aligned} &\underbrace{\left[(1+x)y^2 - e^x\right]}\,dx + (2xy)\,dy = 0 \\ &\frac{\partial}{\partial y}\left[\quad\right] \neq \frac{\partial}{\partial x}(2xy) \end{aligned}\right\}\ \text{inexact.}$$

also not separable or linear.

$$2xy\,\frac{dy}{dx} + (1+x)\,y^2 = e^x$$

$$\boxed{2xy\,\frac{dy}{dx} + y^2} = e^x - xy^2$$

Notice: $\boxed{\frac{d}{dx}\left[xy^2\right] = \frac{d}{dx}[x]\,y^2 + x\,\frac{d}{dx}\left[y^2\right]}$ (Prod.)

$$= y^2 + x\,\frac{d}{dy}\left[y^2\right]\frac{dy}{dx} \qquad \text{(chain)}$$

$$= \boxed{y^2 + 2xy\,\frac{dy}{dx}}$$

The ODE becomes: $\frac{d}{dx}\left[xy^2\right] = e^x - xy^2.$

Let $u(x) := xy^2$. Then $\frac{du}{dx} = e^x - u$

Ex (4), ct'd.

$$\frac{du}{dx} + u = e^x .$$

Solve by int. factor.

- $\mu(x) := e^x$

- $\frac{d}{dx}\left[ue^x\right] = LHS$

- $ue^x = \int e^{2x}\,dx$

$$= \frac{1}{2}e^{2x} + c$$

- $u(x) = \frac{1}{2}e^x + ce^{-x}$

Back-subst. : $xy^2 = \frac{1}{2}e^x + ce^{-x}$ gives an implicit sol'n of the ODE :

$$2xy\frac{dy}{dx} + (1+x)y^2 = e^x .$$

Check : $\frac{d}{dx}\left[xy^2\right] = \frac{d}{dx}\left[\frac{1}{2}e^x + ce^{-x}\right]$

$$\boxed{2xy\frac{dy}{dx} + y^2 = \frac{1}{2}e^x - ce^{-x}}$$

So the LHS of ODE: $2xy\frac{dy}{dx} + y^2 + xy^2 = \boxed{2xy\frac{dy}{dx} + y^2} + \frac{1}{2}e^x + Ce^{-x}$

$$= \frac{1}{2}e^x - ce^{-x} + \frac{1}{2}e^x + ce^{-x}$$

$$= e^x .$$ So ODE is solved.