Ex. 12, prob. 1 (p. 103)

1

$$2xy\frac{dy}{dx} + y^2 = e^x - xy^2$$

Notice: $\frac{d}{dx}[xy^2] = 2xy\frac{dy}{dx} + y^2$

$$\frac{du}{dx} = e^x - u.$$ Solvable by integrating factor.

2

$$\cos y \frac{dy}{dx} + \sin y = x^2.$$

$$\frac{d}{dx}[\sin y] = \cos y \frac{dy}{dx}$$ (chain rule)

$$\sin(y) + \frac{d}{dx}[\sin y] = x^2.$$

Let $u := \sin(y)$

$$u + \frac{du}{dx} = x^2.$$ Linear in u — so can solve by int. factor.

Let $\mu(x) := e^{\int 1\,dx} = e^x$.

$$e^x\frac{du}{dx} + e^x u = e^x x^2$$

$$\frac{d}{dx}[e^x u] = e^x x^2$$

Prob. 2 , p. 103 (cont'd)

$$\int \frac{d}{dx}\left[e^x u\right] dx = \int e^x x^2 \, dx$$

$$e^x u = \int e^x x^2 \, dx \qquad \int u\,dv = uv - \int v\,du$$

Let $u := x^2 \qquad v = e^x$

$du = 2x\,dx \qquad dv := e^x dx$

$$e^x u = x^2 e^x - \int e^x \cdot 2x\,dx$$

$$= x^2 e^x - 2\int x e^x \, dx \qquad \int u\,dv = uv - \int v\,du$$

Let $u := x \qquad v = e^x$

$du = dx \qquad dv := e^x dx$.

$$e^x u = x^2 e^x - 2\left(x e^x - \int e^x dx\right)$$

$$= x^2 e^x - 2\left(x e^x - e^x\right) + c$$

$$= x^2 e^x - 2x e^x + 2e^x + c$$

$$e^x u = e^x\left(x^2 - 2x + 2\right) + c$$

$$u = x^2 - 2x + 2 + ce^{-x}.$$

$u = \sin(y) \Rightarrow y = \arcsin(u)$.

$y = \arcsin\left(x^2 - 2x + 2 + ce^{-x}\right)$

Prob. 2, p. 103, cont'd

Check: $y = \arcsin(x^2 - 2x + 2 + ce^{-x})$

solves $\cos(y)\frac{dy}{dx} + \sin(y) = x^2$.

$\sin(y) = x^2 - 2x + 2 + ce^{-x}$

$\cos(y)\frac{dy}{dx} = \frac{d}{dx}\left[x^2 - 2x + 2 + ce^{-x}\right]$

$= 2x - 2 - ce^{-x}$

So $\cos(y)\frac{dy}{dx} + \sin(y) = (\cancel{2x} \cancel{-2} \cancel{-ce^{-x}}) + (x^2 \cancel{-2x} \cancel{+2} \cancel{+ce^{-x}})$

$= x^2$.

So $\sin(y) = x^2 - 2x + 2 + ce^{-x}$

solves the ODE.

Prob. 7, Ex. 12 (p. 103)

$$x\frac{dy}{dx} - y = \sqrt{x^2+y^2}.$$

Recall:
$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}\left(xy^{-1}\right)$$
$$= y^{-1} + x\frac{d}{dx}\left(y^{-1}\right) \qquad \text{Prod.}$$
$$= y^{-1} + x\frac{d}{dy}\left(y^{-1}\right)\frac{dy}{dx} \qquad \text{Chain}$$
$$= \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}$$

$$-\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{-y + x\frac{dy}{dx}}{y^2}$$

$y>0$

$$\frac{x\frac{dy}{dx} - y}{y^2} = \frac{1}{y^2}\sqrt{x^2+y^2} = \frac{1}{y}\sqrt{\frac{1}{y^2}}\sqrt{x^2+y^2}$$

$$-\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{1}{y}\sqrt{\frac{x^2}{y^2}+1}$$

$$-\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{x}{y}\left(\frac{1}{x}\right)\sqrt{\left(\frac{x}{y}\right)^2+1}$$

Let $u := \frac{x}{y}$.

$$-\frac{du}{dx} = \frac{u}{x}\sqrt{u^2+1} \qquad \text{Separable.}$$

Prob. 7, ctd

$$\frac{-1}{u\sqrt{u^2+1}} \frac{du}{dx} = \frac{1}{x} \qquad \text{separated.}$$

$$\int \frac{-1}{u\sqrt{u^2+1}} \frac{du}{dx}\, dx = \int \frac{1}{x}\, dx$$

$$\int \frac{-1}{u\sqrt{u^2+1}}\, du = \ln|x| + c.$$

$\int \frac{-1}{u\sqrt{u^2+1}}\, du$ can be solved by trig. sub; recall that $\sin^2\theta + \cos^2\theta = 1$, so $\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$,

i.e., $\tan^2\theta + 1 = \sec^2\theta$.

Let $u := \tan\theta$, so $1 + u^2 = 1 + \tan^2\theta = \sec^2\theta$, and

$$\frac{du}{d\theta} = \frac{d}{d\theta}\left[\tan\theta\right] = \frac{d}{d\theta}\left[\frac{\sin\theta}{\cos\theta}\right] = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \sec^2\theta.$$

Then
$$\int \frac{-1}{u\sqrt{u^2+1}}\, du = \int \frac{-\sec^2\theta\, d\theta}{\tan\theta\sqrt{\sec^2\theta}} = \int \frac{-\sec^2\theta\, d\theta}{\tan\theta\sec\theta} =$$

$$= \int \frac{-\sec\theta\, d\theta}{\tan\theta} = \int \frac{-d\theta/\cos\theta}{\sin\theta/\cos\theta}$$

So $\displaystyle\int \frac{-1}{u\sqrt{u^2+1}}\,du = \int \frac{-\,d\theta}{\sin\theta}$.

Notice: $\displaystyle\frac{d}{d\theta}\left[\frac{1+\cos\theta}{\sin\theta}\right] = \frac{\sin\theta \frac{d}{d\theta}[1+\cos\theta] - (1+\cos\theta)\frac{d}{d\theta}[\sin\theta]}{\sin^2\theta}$ (Quot. Rule)

$$= \frac{\sin\theta(-\sin\theta) - (1+\cos\theta)\cos\theta}{\sin^2\theta}$$

$$= \frac{-(\cos^2\theta + \sin^2\theta) - 1\cdot\cos\theta}{\sin^2\theta}$$

$$= \frac{-1-\cos\theta}{\sin^2\theta}$$

$$= \frac{-1}{\sin\theta}\left(\frac{1+\cos\theta}{\sin\theta}\right).$$

So multiply integrand by $\left(\dfrac{1+\cos\theta}{\sin\theta}\Big/\dfrac{1+\cos\theta}{\sin\theta}\right)$:

$$\int \frac{-\,d\theta}{\sin\theta} = \int \frac{-1}{\sin\theta}\left(\frac{1+\cos\theta}{\sin\theta}\right)\left(\frac{1}{\left(\frac{1+\cos\theta}{\sin\theta}\right)}\right)d\theta.$$

Let $w := \dfrac{1+\cos\theta}{\sin\theta}$, so (as shown above)

$dw = -\dfrac{1}{\sin\theta}\,w\,d\theta$

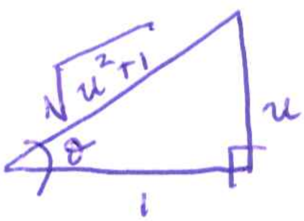
Then $\displaystyle\int \frac{-\,d\theta}{\sin\theta} = \int -\frac{1}{\sin\theta}\,w\left(\frac{1}{w}\right)d\theta = \int \frac{dw}{w} = \ln|w| + C$

So $$\int \frac{-d\theta}{\sin\theta} = \ln|w| + C$$

$$= \ln\left|\frac{1+\cos\theta}{\sin\theta}\right| + C.$$

Then $$\int \frac{-1}{u\sqrt{u^2+1}}\,du = \int \frac{-d\theta}{\sin\theta} = \ln\left|\frac{1+\cos\theta}{\sin\theta}\right| + C.$$

To back-substitute and obtain an expression for u, recall: $u = \tan\theta$ implies the following right triangle:



which permits the observation that

$$\sin\theta = \frac{u}{\sqrt{u^2+1}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{u^2+1}}.$$

Then $$\int \frac{-1}{u\sqrt{u^2+1}}\,du = \ln\left|\frac{1+\cos\theta}{\sin\theta}\right| + C$$

$$= \ln\left|\frac{1 + 1/\sqrt{u^2+1}}{u/\sqrt{u^2+1}}\right| + C$$

$$= \ln\left|\frac{1+\sqrt{u^2+1}}{u}\right| + C$$

So $\displaystyle\int \frac{-1}{u\sqrt{u^2+1}}\,du = \ln\left|\frac{\sqrt{u^2+1}+1}{u}\right| + C$

$$\ln\left|\frac{\sqrt{u^2+1}+1}{u}\right| = \ln|x| + C$$

$$\frac{\sqrt{u^2+1}+1}{u} = e^{\ln|x|+c} = e^{c}e^{\ln|x|} = Ax$$

$$\frac{\sqrt{u^2+1}+1}{u} = Ax$$

If $u = \frac{x}{y}$, then the sol'n becomes

$$\frac{\sqrt{\left(\frac{x}{y}\right)^2+1}+1}{x/y} = Ax$$

$$\sqrt{\left(\frac{x}{y}\right)^2+1}+1 = \frac{Ax^2}{y}.$$

Prob. 7, ct'd.

$$y\sqrt{\left(\frac{x}{y}\right)^2+1} + y = Ax^2$$

$$\sqrt{y^2}\sqrt{\left(\frac{x}{y}\right)^2+1} + y = Ax^2$$

$$\sqrt{y^2\left(\frac{x^2}{y^2}\right)+y^2} + y = Ax^2$$

$$\sqrt{x^2+y^2} + y = Ax^2.$$

This is an eq'n that implicitly defines a solution of the ODE

$$x\frac{dy}{dx} - y = \sqrt{x^2+y^2}.$$

It's easiest (?) to verify by making explicit:

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Prob. 7, ctd.

$$\sqrt{x^2+y^2} = Ax^2 - y$$

$$x^2+y^2 = A^2x^4 - 2Ax^2y + y^2$$

$$x^2 = A^2x^4 - 2Ax^2y$$

$$2Ax^2y = A^2x^4 - x^2$$

$$y = \frac{A^2x^4 - x^2}{2Ax^2} = \frac{Ax^2}{2} - \frac{1}{2A}$$

So $\dfrac{dy}{dx} = Ax$.

Then $$x\frac{dy}{dx} - y = Ax^2 - \left(Ax^2 - \sqrt{x^2+y^2}\right)$$
$$= \sqrt{x^2+y^2}.$$