Oct. 20 : Lesson 63, ct'd.

$\boxed{\text{Exercises } 63\ (5)}$

$$G: \begin{vmatrix} \int_a^b f_1^2\,dx & \int_a^b f_1 f_2\,dx & \cdots & \int_a^b f_1 f_m\,dx \\ \int_a^b f_2 f_1\,dx & \int_a^b f_2^2\,dx & \cdots & \int_a^b f_2 f_m\,dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_m f_1\,dx & \int_a^b f_m f_2\,dx & \cdots & \int_a^b f_m^2\,dx \end{vmatrix}$$

In the two-by-two case (corresponds to determining whether $\{f_1, f_2\}$ is lin. indep.), the Grammian is

$$G(\{f_1, f_2\}) = \begin{vmatrix} \int_a^b f_1^2\,dx & \int_a^b f_1 f_2\,dx \\ \int_a^b f_2 f_1\,dx & \int_a^b f_2^2\,dx \end{vmatrix}$$

The set $\{f_1, \dots, f_m\}$ of fns. defined and cts. on $I := [a,b]$ is linearly $\underline{\underline{\text{dependent}}}$ IF AND ONLY IF $G(\{f_1, \dots, f_m\}) \equiv 0$ on $[a,b]$.

Gramian, cont'd.

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EX. 63(5(a))

$$G(\{x, 2x, 3x\}) = \begin{vmatrix} \int_a^b x^2\,dx & \int_a^b 2x^2\,dx & \int_a^b 3x^2\,dx \\ \int_a^b 2x^2\,dx & \int_a^b 4x^2\,dx & \int_a^b 6x^2\,dx \\ \int_a^b 3x^2\,dx & \int_a^b 6x^2\,dx & \int_a^b 9x^2\,dx \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{1}{3}(b^3-a^3) & \frac{2}{3}(b^3-a^3) & b^3-a^3 \\ \frac{2}{3}(b^3-a^3) & \frac{4}{3}(b^3-a^3) & 2(b^3-a^3) \\ b^3-a^3 & 2(b^3-a^3) & 3(b^3-a^3) \end{vmatrix}$$

$$= \frac{1}{3}(b^3-a^3)\begin{vmatrix} \frac{4}{3}(b^3-a^3) & 2(b^3-a^3) \\ 2(b^3-a^3) & 3(b^3-a^3) \end{vmatrix} +$$

$$+ (-1)\,\frac{2}{3}(b^3-a^3)\begin{vmatrix} \frac{2}{3}(b^3-a^3) & 2(b^3-a^3) \\ b^3-a^3 & 3(b^3-a^3) \end{vmatrix} +$$

$$+ (b^3-a^3)\begin{vmatrix} \frac{2}{3}(b^3-a^3) & \frac{4}{3}(b^3-a^3) \\ b^3-a^3 & 2(b^3-a^3) \end{vmatrix}$$

$$G = \frac{1}{3}(b^3-a^3)\left[\frac{4}{3}\cdot 3(b^3-a^3)^2 - 4(b^3-a^3)^2\right] -$$

$$- \frac{2}{3}(b^3-a^3)\left[\frac{2}{3}\cdot 3(b^3-a^3)^2 - 2(b^3-a^3)^2\right] +$$

$$+ (b^3-a^3)\left[\underline{\frac{2}{3}\cdot 2(b^3-a^3)^2 - \frac{4}{3}(b^3-a^3)^2}\right]$$

$$= (b^3-a^3)^3\left[\underbrace{\underbrace{\frac{4}{3}-\frac{4}{3}}_{0} \underbrace{-\frac{4}{3}+\frac{4}{3}}_{0} \underbrace{+\frac{4}{3}-\frac{4}{3}}_{0}}_{0}\right]$$

$$= 0.$$

Since $G(\{x, 2x, 3x\}) \equiv 0$ everywhere on $\mathbb{R}$, we know that the set $\{x, 2x, 3x\}$ is linearly <u>dependent</u> for all $x \in \mathbb{R}$.

Lesson 19(B): Why Linear (In)dependence matters.

Recall: The general form of an n^{th} order linear ODE:

$$f_n(x)\frac{d^n y}{dx^n} + f_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + f_3(x)\frac{d^3y}{dx^3} + f_2(x)\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_0(x)\,y(x) = Q(x)$$

TH'M. The n^{th} ord. linear ODE ( ) with initial cond'ns

$$y(x_0) = y_0,\quad \left.\frac{dy}{dx}\right|_{x_0} = y_1,\quad \left.\frac{d^2y}{dx^2}\right|_{x_0} = y_2,\quad \ldots,\quad \left.\frac{d^ny}{dx^n}\right|_{x_0} = y_n,$$

for $x_0 \in I$ and for $y_0, y_1, \ldots, y_n$ constants,

has one and only one solution.

Without the initial cond'ns ; and a homogeneous eq'n

$$f_m y^{(m)} + f_{m-1} y^{(m-1)} + \cdots + f_2 y'' + f_1 y' + f_0 y = 0 .$$

This eq'n has m many linearly indep. sol'ns.

If we call these sol'ns $y_1, y_2, y_3, \ldots, y_m$*, then for any constants $c_1, c_2, \ldots, c_m \in \mathbb{R}$, the linear combination

$$\boxed{\underbrace{y_h(x)}_{\text{"homogeneous"}} :=: \underbrace{y_c(x)}_{\text{"complementary"}} := c_1 y_1 + c_2 y_2 + \cdots + c_m y_m}$$

solves the homogeneous ODE.

If y_1 solves the ODE, then

$$f_m y_1^{(m)} + f_{m-1} y_1^{(m-1)} + \cdots + f_2 y_1'' + f_1 y_1' + f_0 y_1 = 0$$

If y_2 solves, then

$$f_m y_2^{(m)} + f_{m-1} y_2^{(m-1)} + \cdots + f_2 y_2'' + f_1 y_2' + f_0 y_2 = 0$$

So, let's compute the LHS of the ODE with $y = y_h$ (or y_c)

$$f_m y_h^{(m)} + f_{m-1} y_h^{(m-1)} + \cdots + f_2 y_h'' + f_1 y_h' + f_0 y_h =$$

$$= f_m \left(c_1 y_1^{(m)} + c_2 y_2^{(m)} + \cdots + c_m y_m^{(m)} \right) + f_{m-1} \left(c_1 y_1^{(m-1)} + c_2 y_2^{(m-1)} + \cdots + c_m y_m^{(m-1)} \right) + \cdots$$

$$\cdots + f_1(c_1 y_1' + c_2 y_2' + \cdots + c_m y_m') + f_0(c_1 y_1 + c_2 y_2 + \cdots + c_m y_m)$$

$$= c_1\left[f_m y_1^{(m)} + f_{m-1} y_1^{(m-1)} + \cdots + f_1 y_1' + f_0 y_1\right] + c_2\left[f_m y_2^{(m)} + f_{m-1} y_2^{(m-1)} + \cdots + f_0 y_2\right] +$$

$$+ \cdots + c_m\left[f_m y_m^{(m)} + f_{m-1} y_m^{(m-1)} + \cdots + f_1 y_m' + f_0 y_m\right]$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_m \cdot 0 \qquad = 0.$$

For the non-homogeneous eq'n:

$$f_m y^{(m)} + \cdots + f_1 y' + f_0 y = Q(x),$$

"particular sol'n"

If $y_p(x)$ solves the non-homog. eq'n and $y_h(x)$ solves its homog. version, then

$$y(x) := y_p(x) + y_h(x)$$

is an m-parameter family of sol'ns of the nonhomogeneous version.

AN IMPORTANT THEOREM.

Lesson 20: Solving the n^{th} ord. linear homog. ODE w/ constant coeffs.

The gen. form of the ——— " ——— is:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0,$$

where the a_i ($i \in [0, n] \cap \mathbb{Z}$) are constants and $a_n \neq 0$.

Q. What if we <u>assume</u> tht. $y(x) := e^{\overset{p}{\cancel{m}}x}$ solves the ODE?

Then

$$a_n \frac{d^n}{dx^n}\left[e^{\overset{p}{\cancel{m}}x}\right] + a_{n-1} \frac{d^{n-1}}{dx^{n-1}}\left[e^{\overset{p}{\cancel{m}}x}\right] + \cdots + a_1 \frac{d}{dx}\left[e^{\overset{p}{\cancel{m}}x}\right] + a_0 e^{\overset{p}{\cancel{m}}x} = 0$$

$$a_n p^n e^{px} + a_{n-1} p^{n-1} e^{px} + \cdots + a_1 p e^{px} + a_0 e^{\overset{p}{\cancel{m}}x} = 0$$

$$e^{px}\left[a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0\right] = 0.$$

This will be satisfied (i.e., $y(x) := e^{px}$ solves ODE) if

$$\underbrace{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0}_{\text{an } n^{th} \text{ degree polynomial — find roots.}} = 0.$$

<u>Recall</u> the "<u>Fundamental Thm. of Algebra</u>":

An n^{th} degree polynomial has at least one, but not more than n many distinct roots in the complex plane.